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# GENERALIZED LIE BIALGEBRAS AND JACOBI STRUCTURES ON LIE GROUPS

BY

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ABSTRACT

We study generalized Lie bialgebroids over a single point, that is, generalized Lie bialgebras and we prove that they can be considered as the infinitesimal invariants of Lie groups endowed with a certain type of Jacobi structure. We also propose a method generalizing the Yang–Baxter equation method to obtain generalized Lie bialgebras. Finally, we classify the compact generalized Lie bialgebras.

## 1. Introduction

A Jacobi structure on a manifold $M$ is defined by a 2-vector $\Lambda$ and a vector field $E$ on $M$ such that $[\Lambda, \Lambda] = 2E \wedge \Lambda$ and $[E, \Lambda] = 0$, where $[,]$ is the Schouten–Nijenhuis bracket [25]. If $(M, \Lambda, E)$ is a Jacobi manifold, the space $C^\infty(M, \mathbb{R})$ can be endowed with the Jacobi bracket, which is a local Lie algebra structure in the sense of Kirillov [17], and conversely, a local Lie algebra structure on $C^\infty(M, \mathbb{R})$ induces a Jacobi structure on $M$ [12, 17]. Jacobi manifolds are natural generalizations of Poisson, contact and locally conformal symplectic manifolds.

Jacobi structures and Lie algebroid structures are closely related. In fact, if $M$ is an arbitrary manifold, the vector bundle $TM \times \mathbb{R} \to M$ possesses a natural Lie algebroid structure and if $M$ is a Jacobi manifold, then the bundle of 1-jets $T^*M \times \mathbb{R} \to M$ admits a Lie algebroid structure [16]. However, as Vaisman proved in [38], the pair $(TM \times \mathbb{R}, T^*M \times \mathbb{R})$ is not a Lie bialgebroid in the sense of Mackenzie and Xu [30] (or Kosmann-Schwarzbach [19]). This is an important

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difference with respect to the Poisson case. Indeed, if $M$ is a Poisson manifold, the vector bundle $T^*M \to M$ is a Lie algebroid [1, 4, 10, 37] and, in addition, if we consider the natural Lie algebroid structure on the dual bundle $TM \to M$, then the pair $(TM, T^*M)$ is a Lie bialgebroid [30].

The above result on the relation between Jacobi structures and Lie bialgebroids and some examples of linear Jacobi structures on vector bundles obtained in [14] motivated the introduction, in [15], of generalized Lie bialgebroids, which are pairs $((A, \phi_0), (A^*, X_0))$, where $A$ is a Lie algebroid over $M$, $\phi_0$ is a 1-cocycle in the Lie algebroid cohomology complex of $A$ with trivial coefficients, $A^*$ is the dual bundle to $A$ which admits a Lie algebroid structure and $X_0$ is a 1-cocycle of $A^*$, the Lie algebroid structures of $A$ and $A^*$ and the 1-cocycles $\phi_0$ and $X_0$ satisfying some compatibility conditions. When $\phi_0$ and $X_0$ are zero, the definition reduces to that of a Lie bialgebroid. If $(M, \Lambda, E)$ is a Jacobi manifold, the pair $((TM \times \mathbb{R}, \phi_0), (T^*M \times \mathbb{R}, X_0))$ is a generalized Lie bialgebroid, where

$$\phi_0 = (0,1) \in \Omega^1(M) \times C^\infty(M, \mathbb{R}) \cong \Gamma(T^*M \times \mathbb{R})$$

and

$$X_0 = (-E, 0) \in \mathfrak{X}(M) \times C^\infty(M, \mathbb{R}) \cong \Gamma(TM \times \mathbb{R}).$$

This result and other relations between generalized Lie bialgebroids and Jacobi structures were proved in [15].

A generalized Lie bialgebroid $((A, \phi_0), (A^*, X_0))$ is a generalized Lie bialgebra if the base space $M$ is a single point or, in other words, if $A$ is a real Lie algebra $\mathfrak{g}$ of finite dimension. In [15], we proved that examples of generalized Lie bialgebras can be obtained from algebraic Jacobi structures on a Lie algebra to be defined in Appendix A. If the 1-cocycles $\phi_0$ and $X_0$ are zero, it is just a Lie bialgebra in the sense of Drinfeld [8].

The aim of this paper is to further study the generalized Lie bialgebras and, more precisely, to prove that the one-to-one correspondence between Lie bialgebras and connected simply connected Poisson Lie groups (see [8, 20, 27, 28, 37]) extends to one between generalized Lie bialgebras and certain types of Jacobi structures on Lie groups. The paper is organized as follows. In Section 2, we recall several definitions and results concerning Jacobi manifolds and generalized Lie bialgebroids which will be used in the rest of the paper. In Section 3, we show that the generalized Lie bialgebras can be considered as the infinitesimal invariants of Lie groups endowed with a certain type of Jacobi structure (see Theorems 3.11 and 3.13). In Section 4, we propose a method for obtaining generalized Lie bialgebras, a generalization of the well-known Yang–Baxter equation method for

Lie bialgebras. As a consequence, we deduce that generalized Lie bialgebras can be obtained from algebraic Jacobi structures on a Lie algebra. These results allow us to describe, in Section 5, several examples. In particular, using algebraic contact or locally conformal symplectic structures, we obtain examples of generalized Lie bialgebras. In Section 6, we describe the structure of a generalized Lie bialgebra $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, X_0))$ such that $\mathfrak{g}$ is a compact Lie algebra and $\phi_0 \neq 0$ or $X_0 \neq 0$. Finally, the paper closes with two Appendices. In the first, we discuss some relations between algebraic Jacobi structures and contact or locally conformal symplectic Lie algebras. In the second, we give a simple proof of the following assertion: if $\mathfrak{h}$ is a compact contact Lie algebra of dimension $\geq 3$, then $\mathfrak{h}$ is isomorphic to $\mathfrak{su}(2)$, and we describe all the algebraic contact structures on $\mathfrak{su}(2)$. These results were used in Section 6.

*Notation:* If $M$ is a differentiable manifold of dimension $n$, we will denote by $C^\infty(M, \mathbb{R})$ the algebra of $C^\infty$ real-valued functions on $M$, by $\Omega^k(M)$ the space of $k$-forms on $M$, by $\mathfrak{X}(M)$ the Lie algebra of vector fields, by $\delta$ the de Rham differential on $\Omega^*(M) = \oplus_k \Omega^k(M)$ and by $[,]$ the Schouten–Nijenhuis bracket ([1, 37]). If $G$ is a Lie group with Lie algebra $\mathfrak{g}$, we will denote by $\mathfrak{e}$ the identity element of $G$, by $L_g$: $G \to G$ (respectively, $R_g$: $G \to G$) the left (respectively, right) translation by $g \in G$, by $Ad$: $G \times \wedge^k\mathfrak{g} \to \wedge^k\mathfrak{g}$ the adjoint action of $G$ on $\wedge^k\mathfrak{g}$ and by $ad$: $\mathfrak{g} \times \wedge^k\mathfrak{g} \to \wedge^k\mathfrak{g}$ the adjoint representation of $\mathfrak{g}$ on $\wedge^k\mathfrak{g}$, that is, $ad = T_\mathfrak{e} Ad$. Moreover, if $s \in \wedge^k\mathfrak{g}$ then $\bar{s}$ (respectively, $\tilde{s}$) is the left (respectively, right) invariant $k$-vector on $G$ defined by $\bar{s}(g) = (L_g)_*(s)$ (respectively, $\tilde{s}(g) = (R_g)_*(s)$), for all $g \in G$, and if $\hat{P}$ is a $k$-vector on $G$ then $\hat{P}_r$: $G \to \wedge^k\mathfrak{g}$ is the map given by

$$\hat{P}_r(g) = (R_{g^{-1}})_*(\hat{P}(g)), \tag{1.1}$$

for all $g \in G$.

## 2. Generalized Lie bialgebroids and Jacobi structures

2.1. Jacobi structures and Lie algebroids. A **Jacobi structure** on $M$ is a pair $(\Lambda, E)$, where $\Lambda$ is a 2-vector and $E$ is a vector field on $M$ satisfying the following properties:

$$[\Lambda, \Lambda] = 2E \wedge \Lambda, \quad [E, \Lambda] = 0. \tag{2.1}$$

The manifold $M$ endowed with a Jacobi structure is called a **Jacobi manifold.** The **Jacobi bracket** of functions is defined by

$$\{f, g\} = \Lambda(\delta f, \delta g) + fE(g) - gE(f), \tag{2.2}$$

for all $f, g \in C^\infty(M, \mathbb{R})$. In fact, the space $C^\infty(M, \mathbb{R})$ endowed with the Jacobi bracket is a **local Lie algebra** in the sense of Kirillov (see [17]). Conversely, a structure of local Lie algebra on $C^\infty(M, \mathbb{R})$ defines a Jacobi structure on $M$ (see [12, 27]). If the vector field $E$ identically vanishes then $(M, \Lambda)$ is a **Poisson manifold**. Jacobi and Poisson manifolds were introduced by Lichnerowicz ([24, 25]) (see also [1, 6, 17, 23, 37, 39]).

A **Lie algebroid** $A$ over a manifold $M$ is a vector bundle $A$ over $M$ together with a Lie bracket $[\![,]\!]$ on the space $\Gamma(A)$ of the global cross sections of $A \to M$ and a bundle map $\rho$: $A \to TM$, called the **anchor map**, such that if we also denote by $\rho$: $\Gamma(A) \to \mathfrak{X}(M)$ the homomorphism of $C^\infty(M, \mathbb{R})$-modules induced by the anchor map then:

(i) $\rho$: $(\Gamma(A), [\![,]\!]) \to (\mathfrak{X}(M), [,])$ is a Lie algebra homomorphism and

(ii) for all $f \in C^\infty(M, \mathbb{R})$ and for all $X, Y \in \Gamma(A)$, one has

$$[\![X, fY]\!] = f[\![X, Y]\!] + (\rho(X)(f))Y.$$

The triple $(A, [\![,]\!], \rho)$ is called a Lie algebroid over $M$ (see [29, 34]).

A real Lie algebra of finite dimension is a Lie algebroid over a point. Another example of a Lie algebroid is the triple $(TM, [,], Id)$, where $M$ is a differentiable manifold and $Id$: $TM \to TM$ is the identity map.

If $A$ is a Lie algebroid, the Lie bracket on $\Gamma(A)$ can be extended to the so-called **Schouten bracket** $[\![,]\!]$ on the space $\Gamma(\wedge^* A) = \oplus_k \Gamma(\wedge^k A)$ of multi-sections of $A$ in such a way that

$$\begin{gathered}
[\![X, f]\!] = \rho(X)(f), \quad [\![P, P']\!] = (-1)^{kk'}[\![P', P]\!], \\
[\![P, P' \wedge P'']\!] = [\![P, P']\!] \wedge P'' + (-1)^{k'(k+1)} P' \wedge [\![P, P'']\!], \\
(-1)^{kk''}[\![[\![P, P']\!], P'']\!] + (-1)^{k'k''}[\![[\![P'', P]\!], P']\!] + (-1)^{kk'}[\![[\![P', P'']\!], P]\!] = 0,
\end{gathered}$$

for $f \in C^\infty(M, \mathbb{R})$, $X \in \Gamma(A), P \in \Gamma(\wedge^k A), P' \in \Gamma(\wedge^{k'} A)$ and $P'' \in \Gamma(\wedge^{k''} A)$.

*Remark 2.1:* The definition of Schouten bracket considered here is the one given in [37] (see also [1, 24]). Some authors, see for example [19], define the Schouten bracket in another way. In fact, the relation between the Schouten bracket $[\![,]\!]'$ in the sense of [19] and the Schouten bracket $[\![,]\!]$ in the sense of [37] is the following one. If $P \in \Gamma(\wedge^k A)$ and $Q \in \Gamma(\wedge^* A)$, then $[\![P, Q]\!]' = (-1)^{k+1}[\![P, Q]\!]$.

On the other hand, imitating the de Rham differential on the space $\Omega^*(M)$, we define the **differential of the Lie algebroid** $A$, $d$: $\Gamma(\wedge^k A^*) \to \Gamma(\wedge^{k+1} A^*)$, as follows. For $\omega \in \Gamma(\wedge^k A^*)$ and $X_0, \dots, X_k \in \Gamma(A)$,

$$\begin{aligned} d\omega(X_0, \dots, X_k) =& \sum_{i=0}^{k} (-1)^i \rho(X_i)(\omega(X_0, \dots, \hat{X}_i, \dots, X_k)) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([\![X_i, X_j]\!], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_k). \end{aligned}$$

Moreover, since $d^2 = 0$, we have the corresponding cohomology spaces. This cohomology is the **Lie algebroid cohomology with trivial coefficients** (see [29]).

Using the above definitions, it follows that a 1-cochain $\phi \in \Gamma(A^*)$ is a 1-cocycle if and only if

$$\phi[\![X, Y]\!] = \rho(X)(\phi(Y)) - \rho(Y)(\phi(X)),$$

for all $X, Y \in \Gamma(A)$.

Next, we will consider two examples of Lie algebroids.

**1.-** *The Lie algebroid* $(TM \times \mathbb{R}, [,], \pi)$

If $M$ is a differentiable manifold, then the triple $(TM \times \mathbb{R}, [,], \pi)$ is a Lie algebroid over $M$, where $\pi\colon TM \times \mathbb{R} \to TM$ is the canonical projection over the first factor and $[,]$ is the bracket given by (see [14, 15, 33, 38])

$$[(X, f), (Y, g)] = ([X, Y], X(g) - Y(f)), \tag{2.3}$$

for $(X, f), (Y, g) \in \mathfrak{X}(M) \times C^\infty(M, \mathbb{R}) \cong \Gamma(TM \times \mathbb{R})$.

**2.-** *The Lie algebroid* $(T^*M \times \mathbb{R}, [\![,]\!]_{(\Lambda,E)}, \widetilde{\#}_{(\Lambda,E)})$ *associated with a Jacobi manifold* $(M, \Lambda, E)$

If $A \to M$ is a vector bundle over $M$ and $P \in \Gamma(\wedge^2 A)$ is a 2-section of $A$, we will denote by $\#_P\colon A^* \to A$ the bundle map given by $\beta(\#_P(\alpha)) = P(x)(\alpha, \beta)$, for $\alpha, \beta \in A^*_x$, $A^*_x$ being the fiber of $A^*$ over $x \in M$. We will also denote by $\#_P\colon \Gamma(A^*) \to \Gamma(A)$ the corresponding homomorphism of $C^\infty(M, \mathbb{R})$-modules. Then, a Jacobi manifold $(M, \Lambda, E)$ has an associated Lie algebroid $(T^*M \times \mathbb{R}, [\![,]\!]_{(\Lambda,E)}, \widetilde{\#}_{(\Lambda,E)})$, where $[\![,]\!]_{(\Lambda,E)}$ and $\widetilde{\#}_{(\Lambda,E)}$ are defined by

$$\begin{aligned} [\![(\alpha, f), (\beta, g)]\!]_{(\Lambda,E)} =& (\mathcal{L}_{\#_\Lambda(\alpha)}\beta - \mathcal{L}_{\#_\Lambda(\beta)}\alpha - \delta(\Lambda(\alpha, \beta)) + f\mathcal{L}_E\beta - g\mathcal{L}_E\alpha \\ &- i(E)(\alpha \wedge \beta), \Lambda(\beta, \alpha) + \#_\Lambda(\alpha)(g) \\ &- \#_\Lambda(\beta)(f) + fE(g) - gE(f)), \\ \widetilde{\#}_{(\Lambda,E)}(\alpha, f) =& \#_\Lambda(\alpha) + fE, \end{aligned} \tag{2.4}$$

for $(\alpha, f), (\beta, g) \in \Omega^1(M) \times C^\infty(M, \mathbb{R})$, $\mathcal{L}$ being the Lie derivative operator (see [16]).

In the particular case when $(M,\Lambda)$ is a Poisson manifold we recover, by projection, the Lie algebroid $(T^*M, [\![,]\!]_\Lambda, \#_\Lambda)$, where $[\![,]\!]_\Lambda$ is the bracket of 1-forms defined by $[\![\alpha,\beta]\!]_\Lambda = \mathcal{L}_{\#_\Lambda(\alpha)}\beta - \mathcal{L}_{\#_\Lambda(\beta)}\alpha - \delta(\Lambda(\alpha,\beta))$, for $\alpha,\beta\in\Omega^1(M)$ (see [1, 4, 10, 37]).

2.2. Generalized Lie bialgebroids. In this Section, we will recall the definition of a generalized Lie bialgebroid. First, we will exhibit some results about the differential calculus on Lie algebroids in the presence of a 1-cocycle (for more details, see [15]).

If $(A, [\![,]\!], \rho)$ is a Lie algebroid over $M$ and, in addition, we have a 1-cocycle $\phi_0 \in \Gamma(A^*)$ then the usual representation of the Lie algebra $\Gamma(A)$ on the space $C^\infty(M,\mathbb{R})$ can be modified and a new representation is obtained. This representation is given by $\rho_{\phi_0}(X)(f) = \rho(X)(f) + \phi_0(X)f$, for $X\in\Gamma(A)$ and $f\in C^\infty(M,\mathbb{R})$. The resulting cohomology operator $d_{\phi_0}$ is called the $\phi_0$-differential of $A$ and its expression, in terms of the differential $d$ of $A$, is $d_{\phi_0}\omega = d\omega + \phi_0\wedge\omega$, for $\omega\in\Gamma(\wedge^k A^*)$. The $\phi_0$-differential of $A$ allows us to define, in a natural way, the $\phi_0$-Lie derivative by a section $X\in\Gamma(A)$, $(\mathcal{L}_{\phi_0})_X\colon \Gamma(\wedge^k A^*)\to\Gamma(\wedge^k A^*)$, as the commutator of $d_{\phi_0}$ and the contraction by $X$, that is, $(\mathcal{L}_{\phi_0})_X = d_{\phi_0}\circ i(X) + i(X)\circ d_{\phi_0}$ (for the general definition of the differential and the Lie derivative associated with a representation of a Lie algebroid on a vector bundle, see [29]).

On the other hand, imitating the definition of the Schouten bracket of two multilinear first-order differential operators on the space of $C^\infty$ real-valued functions on a manifold $N$ (see [1]), we introduced the $\phi_0$-Schouten bracket of a $k$-section $P$ and a $k'$-section $P'$ as the $(k+k'-1)$-section given by

$$(2.5)\quad [\![P,P']\!]_{\phi_0} = [\![P,P']\!] + (-1)^{k+1}(k-1)P\wedge(i(\phi_0)P') - (k'-1)(i(\phi_0)P)\wedge P',$$

where $[\![,]\!]$ is the usual Schouten bracket of $A$. The $\phi_0$-Schouten bracket satisfies the following properties. For $f\in C^\infty(M,\mathbb{R})$, $X,Y\in\Gamma(A)$, $P\in\Gamma(\wedge^k A)$, $P'\in\Gamma(\wedge^{k'}A)$ and $P''\in\Gamma(\wedge^{k''}A)$,

$$[\![X,f]\!]_{\phi_0} = \rho_{\phi_0}(X)(f),\ [\![X,Y]\!]_{\phi_0} = [\![X,Y]\!],\ [\![P,P']\!]_{\phi_0} = (-1)^{kk'}[\![P',P]\!]_{\phi_0},$$
$$[\![P,P'\wedge P'']\!]_{\phi_0} = [\![P,P']\!]_{\phi_0}\wedge P'' + (-1)^{k'(k+1)}P'\wedge[\![P,P'']\!]_{\phi_0} - (i(\phi_0)P)\wedge P'\wedge P'',$$
$$(-1)^{kk''}[\![[\![P,P']\!]_{\phi_0},P'']\!]_{\phi_0} + (-1)^{k'k''}[\![[\![P'',P]\!]_{\phi_0},P']\!]_{\phi_0} + (-1)^{kk'}[\![[\![P',P'']\!]_{\phi_0},P]\!]_{\phi_0} = 0.$$

Using the $\phi_0$-Schouten bracket, we can define the $\phi_0$-Lie derivative of $P\in\Gamma(\wedge^k A)$ by $X\in\Gamma(A)$ as $(\mathcal{L}_{\phi_0})_X(P) = [\![X,P]\!]_{\phi_0}$.

Now, suppose that $(A,[\![,]\!],\rho)$ is a Lie algebroid and that $\phi_0\in\Gamma(A^*)$ is a 1-cocycle. Assume also that the dual bundle $A^*$ admits a Lie algebroid structure

$([\![,]\!]_*, \rho_*)$ and that $X_0 \in \Gamma(A)$ is a 1-cocycle. The pair $((A, \phi_0), (A^*, X_0))$ is a **generalized Lie bialgebroid** if

$$\begin{aligned} d_{*X_0}[\![X, Y]\!] =& [\![X, d_{*X_0}Y]\!]_{\phi_0} - [\![Y, d_{*X_0}X]\!]_{\phi_0}, \\ (\mathcal{L}_{*X_0})_{\phi_0}P +& (\mathcal{L}_{\phi_0})_{X_0}P = 0, \end{aligned} \tag{2.6}$$

for $X, Y \in \Gamma(A)$ and $P \in \Gamma(\wedge^k A)$, where $d_{*X_0}$ (respectively, $\mathcal{L}_{*X_0}$) is the $X_0$-differential (respectively, the $X_0$-Lie derivative) of $A^*$. Note that the second equality in (2.6) holds if and only if $\phi_0(X_0) = 0$, $\rho(X_0) = -\rho_*(\phi_0)$ and $(\mathcal{L}_{*X_0})_{\phi_0}X + [\![X_0, X]\!] = 0$, for $X \in \Gamma(A)$ (for more details, see [15]). Moreover, in the particular case when $\phi_0 = 0$ and $X_0 = 0$, (2.6) is equivalent to the condition $d_*[\![X, Y]\!] = [\![X, d_*Y]\!] - [\![Y, d_*X]\!]$. Thus, the pair $((A, 0), (A^*, 0))$ is a generalized Lie bialgebroid if and only if the pair $(A, A^*)$ is a Lie bialgebroid (see [19, 30]).

On the other hand, if $(M, \Lambda, E)$ is a Jacobi manifold, then we proved in [15] that the pair $\Big((TM \times \mathbb{R}, \phi_0), (T^*M \times \mathbb{R}, X_0)\Big)$ is a generalized Lie bialgebroid, where $\phi_0$ and $X_0$ are the 1-cocycles on $TM \times \mathbb{R}$ and $T^*M \times \mathbb{R}$ given by

$$\begin{aligned} \phi_0 =& (0, 1) \in \Omega^1(M) \times C^\infty(M, \mathbb{R}) \cong \Gamma(T^*M \times \mathbb{R}), \\ X_0 =& (-E, 0) \in \mathfrak{X}(M) \times C^\infty(M, \mathbb{R}) \cong \Gamma(TM \times \mathbb{R}). \end{aligned}$$

*Remark 2.2:* (i) Very recently, an interesting characterization of generalized Lie bialgebroids has been obtained by Grabowski and Marmo [11] as follows. Let $(A, [\![,]\!], \rho)$ be a Lie algebroid and $\phi_0 \in \Gamma(A^*)$ be a 1-cocycle. Assume also that the dual bundle $A^*$ admits a Lie algebroid structure $([\![,]\!]_*, \rho_*)$ and that $X_0 \in \Gamma(A)$ is a 1-cocycle. If we consider the bracket $[\![,]\!]'_{\phi_0}$ of a $k$-section $P$ and a $k'$-section $P'$ as the $(k + k' - 1)$-section given by

$$[\![P, P']\!]'_{\phi_0} = (-1)^{k+1}[\![P, P']\!]_{\phi_0},$$

then $((A, \phi_0), (A^*, X_0))$ is a generalized Lie bialgebroid if and only if $d_{*X_0}$ is a derivation of $(\oplus_k \Gamma(\wedge^k A), [\![,]\!]'_{\phi_0})$, that is,

$$d_{*X_0}[\![P, P']\!]'_{\phi_0} = [\![d_{*X_0}P, P']\!]'_{\phi_0} + (-1)^{k+1}[\![P, d_{*X_0}P']\!]'_{\phi_0}$$

for $P \in \Gamma(\wedge^k A)$ and $P' \in \Gamma(\wedge^* A)$.

(ii) After finishing this paper, we have proved that generalized Lie bialgebroids are particular examples of generalized Lie bialgebras in the sense of Tan and Liu [35]. Several properties of this last type of structures were obtain by Tan and Liu and some interesting examples were given in [35]. However, the particular case of a generalized Lie bialgebroid and its relation with the Jacobi structures was not discussed in this paper.

## 3. Generalized Lie bialgebras, $(\sigma, c)$-multiplicative multivectors and Jacobi structures on Lie groups

In this Section, we will deal with generalized Lie bialgebroids over a point.

*Definition 3.1: [15].* A generalized Lie bialgebra is a generalized Lie bialgebroid over a point, that is, a pair $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, X_0))$, where $(\mathfrak{g}, [,]^{\mathfrak{g}})$ is a real Lie algebra of finite dimension such that the dual space $\mathfrak{g}^*$ is also a Lie algebra with Lie bracket $[,]^{\mathfrak{g}^*}$, $X_0 \in \mathfrak{g}$ and $\phi_0 \in \mathfrak{g}^*$ are 1-cocycles on $\mathfrak{g}^*$ and $\mathfrak{g}$, respectively, and

$$d_{*X_0}[X, Y]^{\mathfrak{g}} = [X, d_{*X_0}Y]^{\mathfrak{g}}_{\phi_0} - [Y, d_{*X_0}X]^{\mathfrak{g}}_{\phi_0}, \tag{3.1}$$

$$\phi_0(X_0) = 0, \tag{3.2}$$

$$i(\phi_0)(d_*X) + [X_0, X]^{\mathfrak{g}} = 0, \tag{3.3}$$

for all $X, Y \in \mathfrak{g}$, $d_*$ being the Chevalley–Eilenberg differential of $(\mathfrak{g}^*, [,]^{\mathfrak{g}^*})$ (acting on $\mathfrak{g} = \wedge^1\mathfrak{g} \subset \wedge^*\mathfrak{g}$).

*Remark 3.2:* In the particular case when $\phi_0 = 0$ and $X_0 = 0$, we recover the concept of a **Lie bialgebra** [8], that is, a pair of Lie algebras in duality $(\mathfrak{g}, \mathfrak{g}^*)$ such that $d_*[X, Y]^{\mathfrak{g}} = [X, d_*Y]^{\mathfrak{g}} - [Y, d_*X]^{\mathfrak{g}}$, for $X, Y \in \mathfrak{g}$ (see [19, 30]).

*Remark 3.3:* (i) It is well-known that Lie bialgebras may be identified with Manin triples of Lie algebras (see [9]). In fact, if $(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie bialgebra, one may define a Lie bracket on the direct sum $\mathfrak{g} \oplus \mathfrak{g}^*$ in such a way that $\mathfrak{g}$ and $\mathfrak{g}^*$ are Lie subalgebras of $\mathfrak{g} \oplus \mathfrak{g}^*$ and $\mathfrak{g}$ and $\mathfrak{g}^*$ are isotropic subspaces of $\mathfrak{g} \oplus \mathfrak{g}^*$ with respect to the natural symmetric pairing. The triple $(\mathfrak{g} \oplus \mathfrak{g}^*, \mathfrak{g}, \mathfrak{g}^*)$ is a Manin triple. Now, suppose that $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, X_0))$ is a generalized Lie bialgebra. Then, following the construction of Manin triples for generalized Lie bialgebras in the sense of Tan and Liu [35], one may introduce an $\mathbb{R}$-bilinear skew-symmetric bracket $[,]^{\mathfrak{g}\oplus\mathfrak{g}^*}$ on the space $\mathfrak{g} \oplus \mathfrak{g}^*$ given by

$$\begin{aligned} &[X \oplus \alpha, Y \oplus \beta]^{\mathfrak{g}\oplus\mathfrak{g}^*} = \\ &\qquad ([X, Y]^{\mathfrak{g}} + (\mathcal{L}_{*X_0})_\alpha Y - (\mathcal{L}_{*X_0})_\beta X - \frac{1}{2}(\alpha(Y) - \beta(X))X_0) \\ &\qquad \oplus ([\alpha, \beta]^{\mathfrak{g}^*} + (\mathcal{L}_{\phi_0})_X\beta - (\mathcal{L}_{\phi_0})_Y\alpha + \frac{1}{2}(\alpha(Y) - \beta(X))\phi_0), \end{aligned} \tag{3.4}$$

for $X \oplus \alpha, Y \oplus \beta \in \mathfrak{g} \oplus \mathfrak{g}^*$. However, using the results in [35], it follows that $(\mathfrak{g} \oplus \mathfrak{g}^*, [,]^{\mathfrak{g}\oplus\mathfrak{g}^*})$ is not, in general, a Lie algebra.

(ii) If $F$ is a finite-dimensional real vector space and $F^*$ is its dual vector space, on the graded vector space $\wedge(F^* \oplus F)$ there exists a graded Lie bracket, called the **big bracket**, which was introduced in [21]. The definition of Lie bialgebras may be formulated, in a very simple form, using this bracket (see [18, 22]). We can also use the big bracket in order to formulate the definition of generalized Lie bialgebras. However, this does not clarify and simplify the compatibility conditions. Perhaps (see Remark 2.2(i)), it could be more interesting to introduce a suitable modification of the big bracket, the **$\phi_0 \oplus X_0$-big bracket** on $\wedge(F^* \oplus F)$ with $\phi_0 \in F^*$ and $X_0 \in F$, and then to formulate the definition of generalized Lie bialgebras in terms of the $\phi_0 \oplus X_0$-big bracket.

The construction of the $\phi_0 \oplus X_0$-big bracket on $\wedge(F^* \oplus F)$ (and its applications) and a detailed study of the properties of the bracket $[,]^{\mathfrak{g}\oplus\mathfrak{g}^*}$ given by (3.4) and its relation with generalized Lie bialgebras will be the subject of a forthcoming paper.

We know that there exists a one-to-one correspondence between Lie bialgebras and connected simply connected Poisson Lie groups (see [8, 27, 28, 37]). So, we will study a connected Lie group $G$ with Lie algebra $\mathfrak{g}$ such that the pair $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, X_0))$ is a generalized Lie bialgebra.

We will use the following well-known results about cocycles on Lie groups and on their Lie algebras.

LEMMA 3.4 ([27]): *Let $G$ be a connected Lie group with Lie algebra $\mathfrak{g}$. Let $\Phi: G \times V \to V$ be a representation of $G$ on a vector space $V$. Let $T_{\mathfrak{e}}\Phi: \mathfrak{g} \times V \to V$ be the induced representation of $\mathfrak{g}$ on $V$.*

(i) *If the map $\phi: G \to V$ is a 1-cocycle on $G$ relative to $\Phi$, i.e., if for $h, g \in G$*

$$\phi(hg) = \phi(h) + \Phi(h, \phi(g)),$$

*then $\epsilon =: (\delta\phi)(\mathfrak{e}): \mathfrak{g} \to V$, the derivative of $\phi$ at $\mathfrak{e}$, is a 1-cocycle on $\mathfrak{g}$ relative to $T_{\mathfrak{e}}\Phi$, i.e., for $X, Y \in \mathfrak{g}$*

$$T_{\mathfrak{e}}\Phi(X, \epsilon(Y)) - T_{\mathfrak{e}}\Phi(Y, \epsilon(X)) = \epsilon([X, Y]^{\mathfrak{g}}).$$

*Moreover, $\delta\phi = 0$ implies that $\phi = 0$.*

(ii) *When $G$ is simply connected, any 1-cocycle $\epsilon$ on $\mathfrak{g}$ relative to $T_{\mathfrak{e}}\Phi$ can be integrated to give a unique 1-cocycle $\phi$ on $G$ relative to $\Phi$ such that $(\delta\phi)(\mathfrak{e}) = \epsilon$.*

(iii) *When $\mathfrak{g}$ is semisimple, every 1-cocycle $\epsilon: \mathfrak{g} \to V$ on $\mathfrak{g}$ is a coboundary, that is, $\epsilon(X) = T_{\mathfrak{e}}\Phi(X, v_0)$, for some $v_0 \in V$.*

Next, we will introduce the notion of a $(\sigma, c)$-multiplicative $k$-vector on a connected Lie group $G$, where $\sigma\colon G \to \mathbb{R}$ is a multiplicative function and $c \in \mathbb{R}$. This notion will play an important role (in Section 3.2) in the description of the Jacobi structure on a connected simply connected Lie group whose Lie algebra is $\mathfrak{g}$ and such that the pair $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, X_0))$ is a generalized Lie bialgebra. We recall that a $C^\infty$ real-valued function $\sigma\colon G \to \mathbb{R}$ is multiplicative if it is a Lie group homomorphism.

3.1. $(\sigma, c)$-MULTIPLICATIVE MULTIVECTORS ON A LIE GROUP. We will denote by $e\colon \mathbb{R} \to \mathbb{R}$ the real exponential. Then,

PROPOSITION 3.5: *Let $G$ be a connected Lie group, $\sigma\colon G \to \mathbb{R}$ a multiplicative function and $c \in \mathbb{R}$. If $\hat{P}$ is a k-vector on $G$, the following properties are equivalent:*

(i) $\hat{P}_r(hg) = \hat{P}_r(h) + e^{-(k-c)\sigma(h)} Ad_h(\hat{P}_r(g))$.

(ii) $\hat{P}(hg) = (R_g)_*(\hat{P}(h)) + e^{-(k-c)\sigma(h)} (L_h)_*(\hat{P}(g))$.

(iii)
$$\begin{aligned} e^{(k-c)\sigma(hg)} \hat{P}(hg) =& e^{(k-c)\sigma(g)} (R_g)_*(e^{(k-c)\sigma(h)} \hat{P}(h)) \\ &+ (L_h)_*(e^{(k-c)\sigma(g)} \hat{P}(g)). \end{aligned}$$

(iv) $\hat{P}(\mathfrak{e}) = 0$ *and* $e^{(k-c)\sigma} \mathcal{L}_{\bar{X}} \hat{P}$ *is left invariant whenever* $\bar{X}$ *is a left invariant vector field on* $G$.

(v) $\hat{P}(\mathfrak{e}) = 0$ *and* $e^{-(k-c)\sigma} \mathcal{L}_{\tilde{X}} (e^{(k-c)\sigma} \hat{P})$ *is right invariant whenever* $\tilde{X}$ *is a right invariant vector field on* $G$.

*Proof:* The result follows using (1.1), the fact that $\sigma$ is a multiplicative function and proceeding as in the proof of Proposition 10.5 in [37]. ∎

Now, we introduce the definition of a $(\sigma, c)$-multiplicative $k$-vector on $G$.

*Definition 3.6:* Let $G$ be a connected Lie group, $\sigma\colon G \to \mathbb{R}$ a multiplicative function and $c \in \mathbb{R}$. A k-vector $\hat{P}$ on $G$ is said to be $(\sigma, c)$-multiplicative if $\hat{P}$ satisfies any of the properties in Proposition 3.5. In particular, if $c = 1$, we will say that the k-vector is $\sigma$-multiplicative.

It is clear that if $\hat{P}$ is a $(\sigma, c)$-multiplicative $k$-vector and $\sigma$ identically vanishes, then $\hat{P}$ is multiplicative (see [27, 37]).

Let $G$ be a connected Lie group with Lie algebra $\mathfrak{g}$, $\sigma\colon G \to \mathbb{R}$ a multiplicative function and $c \in \mathbb{R}$. We can introduce the representation $Ad_{(\sigma,c)}\colon G \times \wedge^k \mathfrak{g} \to \wedge^k \mathfrak{g}$ of $G$ on $\wedge^k \mathfrak{g}$ defined by

$$(Ad_{(\sigma,c)})_g(s) = e^{-(k-c)\sigma(g)} Ad_g s, \tag{3.5}$$

for $g \in G$ and $s \in \wedge^k \mathfrak{g}$. If $\phi_0 = (\delta\sigma)(\mathfrak{e})$ then we will denote by $ad_{(\phi_0,c)}$ the corresponding representation of $\mathfrak{g}$ on $\wedge^k\mathfrak{g}$, that is, $ad_{(\phi_0,c)} = T_{\mathfrak{e}} Ad_{(\sigma,c)}: \mathfrak{g} \times \wedge^k \mathfrak{g} \to \wedge^k \mathfrak{g}$. From (3.5), it follows that

$$(3.6) \quad ad_{(\phi_0,c)}(X)(s) = [X,s]^{\mathfrak{g}} - (k-c)\phi_0(X)s = ad(X)(s) - (k-c)\phi_0(X)s$$

for $X \in \mathfrak{g}$ and $s \in \wedge^k \mathfrak{g}$, where $[,]^{\mathfrak{g}}$ is the Schouten bracket of the Lie algebroid $\mathfrak{g} \to \{$ a single point$\}$. It is clear that $\phi_0 \in \mathfrak{g}^*$ is a 1-cocycle with respect to the trivial representation of $\mathfrak{g}$ on $\mathbb{R}$ and that if $c = 1$ then (see (2.5))

$$(3.7) \qquad ad_{(\phi_0,1)}(X)(s) = [X,s]^{\mathfrak{g}}_{\phi_0}.$$

*Remark 3.7:* Note that if $\hat{P}$ is a $k$-vector on $G$ then, from (3.5) and Proposition 3.5, we obtain that $\hat{P}$ is $(\sigma,c)$-multiplicative if and only if $\hat{P}_r: G \to \wedge^k\mathfrak{g}$ is a 1-cocycle with respect to the representation $Ad_{(\sigma,c)}: G \times \wedge^k\mathfrak{g} \to \wedge^k\mathfrak{g}$.

Now, suppose that $\hat{P}$ is a $k$-vector on $G$ such that $\hat{P}(\mathfrak{e}) = 0$. Then, one can define the intrinsic derivative of $\hat{P}$ at $\mathfrak{e}$ as the linear map $\delta_{\mathfrak{e}}\hat{P}: \mathfrak{g} \to \wedge^k\mathfrak{g}$ given by (see [27, 37])

$$(3.8) \qquad (\delta_{\mathfrak{e}}\hat{P})(X) = (\delta\hat{P}_r)(\mathfrak{e})(X) = (\mathcal{L}_{\hat{X}}\hat{P})(\mathfrak{e}),$$

for $X \in \mathfrak{g}$, $\hat{X}$ being an arbitrary vector field on $G$ satisfying $\hat{X}(\mathfrak{e}) = X$. Using (3.8), Lemma 3.4 and Remark 3.7, we deduce

PROPOSITION 3.8: *Let $G$ be a connected Lie group, $\sigma: G \to \mathbb{R}$ a multiplicative function and $c \in \mathbb{R}$. Suppose that $\phi_0 = (\delta\sigma)(\mathfrak{e})$.*

(i) *If $\hat{P}$ is a $(\sigma,c)$-multiplicative $k$-vector then its intrinsic derivative $\delta_{\mathfrak{e}}\hat{P}: \mathfrak{g} \to \wedge^k\mathfrak{g}$ is a 1-cocycle with respect to the representation $ad_{(\phi_0,c)}: \mathfrak{g}\times\wedge^k\mathfrak{g} \to \wedge^k\mathfrak{g}$.*
(ii) *If $G$ is simply connected and $\epsilon: \mathfrak{g} \to \wedge^k\mathfrak{g}$ is a 1-cocycle with respect to the representation $ad_{(\phi_0,c)}: \mathfrak{g} \times \wedge^k\mathfrak{g} \to \wedge^k\mathfrak{g}$ then there exists a unique $(\sigma,c)$-multiplicative $k$-vector $\hat{P}$ such that its intrinsic derivative at $\mathfrak{e}$, $\delta_{\mathfrak{e}}\hat{P}$, is just $\epsilon$.*

*Remark 3.9:* Let $G$ be a connected Lie group, $\sigma: G \to \mathbb{R}$ a multiplicative function and $c \in \mathbb{R}$. If $\hat{P}$ is a $(\sigma,c)$-multiplicative $k$-vector then, from Proposition 3.5, it follows that

$$(\delta\hat{P}_r)(h)((L_h)_*(X)) = e^{-(k-c)\sigma(h)}Ad_h((\delta_{\mathfrak{e}}\hat{P})(X)),$$

for $h \in G$ and $X \in \mathfrak{g}$. Thus, $\delta\hat{P}_r = 0$ if and only if the intrinsic derivative of $\hat{P}$ at $\mathfrak{e}$ is zero. Therefore, $\hat{P} = 0$ if and only if the intrinsic derivative of $\hat{P}$ at $\mathfrak{e}$ is zero (see Lemma 3.4 and Remark 3.7).

*Example 3.10:* Let $G$ be a connected Lie group with Lie algebra $\mathfrak{g}$, $\sigma\colon G \to \mathbb{R}$ a multiplicative function and $c \in \mathbb{R}$. Suppose that $s \in \wedge^k \mathfrak{g}$. Then, we consider the $k$-vector $\hat{s}$ on $G$ defined by

$$\hat{s}(g) = e^{-(k-c)\sigma(g)}\bar{s}(g) - \tilde{s}(g), \quad \text{for all } g \in G.$$

A direct computation shows that $\hat{s}$ is a $(\sigma, c)$-multiplicative $k$-vector on $G$. Moreover, the intrinsic derivative of $\hat{s}$ at $\mathfrak{e}$ is given by

$$(\delta_{\mathfrak{e}}\hat{s})(X) = [X, s]^{\mathfrak{g}} - (k-c)\phi_0(X)s = ad_{(\phi_0,c)}(X)(s),$$

for $X \in \mathfrak{g}$, where $\phi_0 = (\delta\sigma)(\mathfrak{e})$. Note that, in this case, $\delta_{\mathfrak{e}}\hat{s}$ is a 1-coboundary with respect to the representation $ad_{(\phi_0,c)}\colon \mathfrak{g} \times \wedge^k\mathfrak{g} \to \wedge^k\mathfrak{g}$. Moreover, using Remark 3.9, we deduce that $s$ is $ad_{(\phi_0,c)}$-invariant if and only if $\tilde{s} = e^{-(k-c)\sigma}\bar{s}$.

### 3.2. Generalized Lie bialgebras and Jacobi structures on connected Lie groups.

We will prove that if $G$ is a connected simply connected Lie group with Lie algebra $\mathfrak{g}$ and the pair $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, X_0))$ is a generalized Lie bialgebra then $G$ admits a special Jacobi structure.

THEOREM 3.11: *Let $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, X_0))$ be a generalized Lie bialgebra and $G$ a connected simply connected Lie group with Lie algebra $\mathfrak{g}$. Then, there exists a unique multiplicative function $\sigma\colon G \to \mathbb{R}$ and a unique $\sigma$-multiplicative 2-vector $\Lambda$ on $G$ such that $(\delta\sigma)(\mathfrak{e}) = \phi_0$ and the intrinsic derivative of $\Lambda$ at $\mathfrak{e}$ is $-d_{*X_0}$. Moreover, the following relation holds,*

$$\#_\Lambda(\delta\sigma) = \tilde{X}_0 - e^{-\sigma}\bar{X}_0, \tag{3.9}$$

*and the pair $(\Lambda, E)$ is a Jacobi structure on $G$, where $E = -\tilde{X}_0$.*

*Proof:* Since $G$ is connected and simply connected then, using Lemma 3.4 and the fact that $\phi_0$ is a 1-cocycle with respect to the trivial representation of $G$ on $\mathbb{R}$, we deduce that there exists a unique multiplicative function $\sigma\colon G \to \mathbb{R}$ satisfying $(\delta\sigma)(\mathfrak{e}) = \phi_0$. Now, take $\epsilon\colon \mathfrak{g} \to \wedge^2\mathfrak{g}$ given by $\epsilon(X) = -d_{*X_0}X$, for $X \in \mathfrak{g}$. From (3.1), it follows that $\epsilon$ is a 1-cocycle of $\mathfrak{g}$ with respect to the representation $ad_{(\phi_0,1)}$. Thus, by Proposition 3.8, there exists a unique $\sigma$-multiplicative 2-vector $\Lambda$ on $G$ such that its intrinsic derivative at $\mathfrak{e}$ is $-d_{*X_0}$, that is,

$$\delta_{\mathfrak{e}}\Lambda = -d_{*X_0}. \tag{3.10}$$

Next, we will see that (3.9) holds. Let $\bar{X} \in \mathfrak{X}(G)$ be a left invariant vector field and $X = \bar{X}(\mathfrak{e})$. Then, $\mathcal{L}_{\bar{X}}(\delta\sigma) = \delta(\mathcal{L}_{\bar{X}}\sigma) = 0$ and

$$[\bar{X}, \#_\Lambda(\delta\sigma) + e^{-\sigma}\bar{X}_0] = i(\delta\sigma)(\mathcal{L}_{\bar{X}}\Lambda) - e^{-\sigma}\phi_0(X)\bar{X}_0 + e^{-\sigma}\overline{[X, X_0]^{\mathfrak{g}}}.$$

The 2-vector $e^{\sigma}\mathcal{L}_{\bar{X}}\Lambda$ is left invariant (see Proposition 3.5). Therefore, if $g \in G$ and $\alpha_g \in T_g^*G$, we obtain that

$$\begin{aligned}\alpha_g&\Big\{i(\delta\sigma)(\mathcal{L}_{\bar{X}}\Lambda) - e^{-\sigma}\phi_0(X)\bar{X}_0 + e^{-\sigma}\overline{[X,X_0]^{\mathfrak{g}}}\Big\}(g)\\ &=e^{-\sigma(g)}\Big((\mathcal{L}_{\bar{X}}\Lambda)_{(\mathfrak{e})}(((L_g)_*)^*((\delta\sigma)(g)),((L_g)_*)^*\alpha_g)\\ &\quad - \phi_0(X)(((L_g)_*)^*(\alpha_g))(X_0) + (((L_g)_*)^*(\alpha_g))([X,X_0]^{\mathfrak{g}})\Big),\end{aligned}$$

where $((L_g)_*)^*\colon T_g^*G \to \mathfrak{g}^*$ is the adjoint homomorphism of $(L_g)_*\colon \mathfrak{g} \to T_gG$.

Note that the 1-form $\delta\sigma$ is left invariant which implies that $((L_g)_*)^*$ $(\delta\sigma(g))$ $= \phi_0$. Consequently, from (3.2), (3.3), (3.8) and (3.10), we deduce that

$$\alpha_g\Big\{i(\delta\sigma)(\mathcal{L}_{\bar{X}}\Lambda) - e^{-\sigma}\phi_0(X)\bar{X}_0 + e^{-\sigma}\overline{[X,X_0]^{\mathfrak{g}}}\Big\}(g) = 0.$$

Thus, $\#_{\Lambda}(\delta\sigma) + e^{-\sigma}\bar{X}_0$ is a right invariant vector field and, since

$$\Big(\#_{\Lambda}(\delta\sigma) + e^{-\sigma}\bar{X}_0\Big)(\mathfrak{e}) = X_0,$$

we conclude that (3.9) holds.

Now, take $E = -\tilde{X}_0$. Since $E$ is a right invariant vector field and $\Lambda$ is $\sigma$-multiplicative, we have that $e^{-\sigma}\mathcal{L}_E(e^{\sigma}\Lambda)$ is right invariant (see Proposition 3.5). Moreover, from (3.2),

$$e^{-\sigma}\mathcal{L}_E(e^{\sigma}\Lambda) = e^{-\sigma}(e^{\sigma}E(\sigma)\Lambda + e^{\sigma}\mathcal{L}_E\Lambda) = \mathcal{L}_E\Lambda.$$

On the other hand, using (3.8), (3.10) and the fact that $X_0 \in \mathfrak{g}$ is a 1-cocycle (that is, $d_*X_0 = 0$), it follows that $(\mathcal{L}_E\Lambda)(\mathfrak{e}) = 0$. This implies that $\mathcal{L}_E\Lambda = [E,\Lambda] = 0$.

Finally, we will prove that $[\Lambda,\Lambda] - 2E\wedge\Lambda = 0$. First, we will show that $[\Lambda,\Lambda] - 2E\wedge\Lambda$ is $\sigma$-multiplicative. Since $\Lambda(\mathfrak{e}) = 0$, we have that $([\Lambda,\Lambda] - 2E\wedge\Lambda)(\mathfrak{e}) = 0$ (see [37]). Moreover, if $\bar{X}$ is a left invariant vector field then $[\bar{X},E] = 0$ and, using (3.9) and the properties of the Schouten–Nijenhuis bracket, we deduce that

$$e^{2\sigma}\mathcal{L}_{\bar{X}}\Big([\Lambda,\Lambda] - 2E\wedge\Lambda\Big) = 2\Big(e^{\sigma}[e^{\sigma}\mathcal{L}_{\bar{X}}\Lambda,\Lambda] + \bar{X}_0\wedge(e^{\sigma}\mathcal{L}_{\bar{X}}\Lambda)\Big).$$

On the other hand, from Proposition 3.5, it follows that $e^{\sigma}\mathcal{L}_{\bar{X}}\Lambda$ and $e^{\sigma}[e^{\sigma}\mathcal{L}_{\bar{X}}\Lambda,\Lambda]$ are left invariant multivectors. Therefore, $e^{2\sigma}\mathcal{L}_{\bar{X}}\Big([\Lambda,\Lambda] - 2E\wedge\Lambda\Big)$ is also a left invariant multivector. Consequently, $[\Lambda,\Lambda] - 2E\wedge\Lambda$ is $\sigma$-multiplicative, as we wanted to prove.

Next, we will compute the intrinsic derivative at $\mathfrak{e}$ of the 3-vector $[\Lambda,\Lambda]-2E\wedge\Lambda$.

If $[,]_\Lambda\colon \wedge^2 \mathfrak{g}^* \to \mathfrak{g}^*$ is the adjoint map of the intrinsic derivative of $\Lambda$ at $\mathfrak{e}$, using (3.10), we obtain that

$$[\alpha,\beta]_\Lambda = [\alpha,\beta]^{\mathfrak{g}^*} - i(X_0)(\alpha\wedge\beta), \tag{3.11}$$

for $\alpha,\beta\in\mathfrak{g}^*$, where $[,]^{\mathfrak{g}^*}$ is the Lie bracket on $\mathfrak{g}^*$. This implies that

$$[\alpha,\beta]_\Lambda(X_0) = [\alpha,\beta]^{\mathfrak{g}^*}(X_0) = 0. \tag{3.12}$$

Now, from (3.8), (3.10) and since $E = -\tilde{X}_0$, we have that

$$\begin{aligned}\delta_{\mathfrak{e}}([\Lambda,\Lambda] - 2E\wedge\Lambda)(X) =& \mathcal{L}_{\bar{X}}([\Lambda,\Lambda] - 2E\wedge\Lambda)(\mathfrak{e})\\ =&\delta_{\mathfrak{e}}[\Lambda,\Lambda](X) - 2X_0\wedge d_*X,\end{aligned} \tag{3.13}$$

for $X\in\mathfrak{g}$. Thus, using (3.11), (3.12) and (3.13), we conclude that

$$\Big\{\delta_{\mathfrak{e}}([\Lambda,\Lambda] - 2E\wedge\Lambda)(X)\Big\}(\alpha,\beta,\gamma) = -2\sum_{Cycl.(\alpha,\beta,\gamma)}\Big([\alpha,[\beta,\gamma]^{\mathfrak{g}^*}]^{\mathfrak{g}^*}\Big)(X) = 0,$$

for $\alpha,\beta,\gamma\in\mathfrak{g}^*$, that is, the intrinsic derivative of $[\Lambda,\Lambda] - 2E\wedge\Lambda$ at $\mathfrak{e}$ is null. Therefore, $[\Lambda,\Lambda] = 2E\wedge\Lambda$ (see Remark 3.9). ∎

*Remark 3.12:*

(i) Under the same hypotheses as in Theorem 3.11, if $\phi_0 = 0$ then the multiplicative function $\sigma$ will vanish, the 2-vector $\Lambda$ will be multiplicative and the vector field $E$ will be bi-invariant (see (3.9)).

(ii) Under the same hypotheses as in Theorem 3.11, if $\phi_0 = 0$ and $X_0 = 0$ then $\sigma$ and $E$ will be null and $(G,\Lambda)$ will be a Poisson Lie group.

Now, we discuss a converse of Theorem 3.11.

THEOREM 3.13: *Let $(\Lambda, E)$ be a Jacobi structure on a connected Lie group $G$ and $\sigma\colon G\to\mathbb{R}$ a multiplicative function such that:*

(i) *$\Lambda$ is $\sigma$-multiplicative.*

(ii) *$E$ is a right invariant vector field, $E(\mathfrak{e}) = -X_0$ and $\#_\Lambda(\delta\sigma) = \tilde{X}_0 - e^{-\sigma}\bar{X}_0$.*

*If $[,]_\Lambda\colon \wedge^2\mathfrak{g}^*\to\mathfrak{g}^*$ is the adjoint map of the intrinsic derivative of $\Lambda$ at $\mathfrak{e}$ and $[,]^{\mathfrak{g}^*}$ is the bracket on $\mathfrak{g}^*$ given by*

$$[\alpha,\beta]^{\mathfrak{g}^*} = [\alpha,\beta]_\Lambda + i(X_0)(\alpha\wedge\beta), \quad \textit{for } \alpha,\beta\in\mathfrak{g}^*, \tag{3.14}$$

*then $(\mathfrak{g}^*, [,]^{\mathfrak{g}^*})$ is a Lie algebra and the pair $((\mathfrak{g},\phi_0),(\mathfrak{g}^*,X_0))$ is a generalized Lie bialgebra, where $\phi_0 = (\delta\sigma)(\mathfrak{e})$.*

*Proof:* Since $\sigma$ is a multiplicative function, we have that $\phi_0$ is a 1-cocycle of $(\mathfrak{g}, [,]^{\mathfrak{g}})$. Now, suppose that $\alpha, \beta \in \mathfrak{g}^*$. We consider two $C^\infty$ real-valued functions $f$ and $g$ on $G$ such that

$$(3.15) \qquad f(\mathfrak{e}) = g(\mathfrak{e}) = 0, \quad (\delta f)(\mathfrak{e}) = \alpha, \quad (\delta g)(\mathfrak{e}) = \beta.$$

If $\{,\}$ is the Jacobi bracket associated with the Jacobi structure $(\Lambda, E)$ then, from (2.2), (3.8), (3.14) and (3.15), we deduce that

$$(3.16) \qquad (\delta\{f,g\})(\mathfrak{e}) = \delta(\Lambda(\delta f, \delta g))(\mathfrak{e}) + i(X_0)(\alpha \wedge \beta) = [\alpha, \beta]^{\mathfrak{g}^*}.$$

Using (3.16) it follows that $(\mathfrak{g}^*, [,]^{\mathfrak{g}^*})$ is a Lie algebra. Moreover, from (3.14), we obtain that

$$(3.17) \qquad (\delta_{\mathfrak{e}}\Lambda)(X) = -d_{*X_0}X, \quad \text{for } X \in \mathfrak{g}.$$

Thus, using (2.1), (3.8) and (3.17), we prove that $d_* X_0 = 0$, that is, $X_0$ is a 1-cocycle of $(\mathfrak{g}^*, [,]^{\mathfrak{g}^*})$.

On the other hand, since $\Lambda$ is $\sigma$-multiplicative, we conclude that

$$\epsilon = -d_{*X_0} \colon \mathfrak{g} \to \wedge^2 \mathfrak{g}$$

is a 1-cocycle with respect to the representation $ad_{(\phi_0,1)} \colon \mathfrak{g} \times \wedge^2\mathfrak{g} \to \wedge^2\mathfrak{g}$ (see (3.17) and Proposition 3.8). Therefore, (3.1) holds.

Now, the equality $\#_\Lambda(\delta\sigma) = \tilde{X}_0 - e^{-\sigma}\bar{X}_0$ implies that $e^{-\sigma}\bar{X}_0(\sigma) = \tilde{X}_0(\sigma)$, i.e., $e^{-\sigma}X_0(\sigma) = X_0(\sigma)$ (note that $\sigma$ is a multiplicative function). Consequently,

$$(3.18) \qquad \phi_0(X_0) = X_0(\sigma) = 0.$$

Using again that $\#_\Lambda(\delta\sigma) = \tilde{X}_0 - e^{-\sigma}\bar{X}_0$ and the fact that $\sigma$ is a multiplicative function, we obtain that

$$0 = \mathcal{L}_{\bar{X}}(i(\delta\sigma)(\Lambda) + e^{-\sigma}\bar{X}_0) = i(\delta\sigma)(\mathcal{L}_{\bar{X}}\Lambda) - e^{-\sigma}\bar{X}(\sigma)\bar{X}_0 + e^{-\sigma}[\bar{X}, \bar{X}_0]$$

for $X \in \mathfrak{g}$. In particular,

$$(3.19) \qquad \begin{aligned} 0 &= \Big\{i(\delta\sigma)(\mathcal{L}_{\bar{X}}\Lambda) - e^{-\sigma}\bar{X}(\sigma)\bar{X}_0 + e^{-\sigma}[\bar{X}, \bar{X}_0]\Big\}(\mathfrak{e}) \\ &= i(\phi_0)((\delta_{\mathfrak{e}}\Lambda)(X)) - \phi_0(X)X_0 - [X_0, X]^{\mathfrak{g}}. \end{aligned}$$

Thus, from (3.17), (3.18) and (3.19), it follows that $i(\phi_0)(d_* X) + [X_0, X]^{\mathfrak{g}} = 0$. ∎

*Remark 3.14:* If $(\Lambda, E)$ is a Jacobi structure on a connected Lie group $G$ which satisfies the hypotheses of Theorem 3.13, $([\![\, ,\,]\!]_{(\Lambda,E)}, \widetilde{\#}_{(\Lambda,E)})$ is the Lie algebroid structure on $T^*G \times \mathbb{R}$ given by (2.4) and $\alpha, \beta \in \mathfrak{g}^*$, then a direct computation shows that

$$[\![(\hat{\alpha}, f), (\hat{\beta}, g)]\!]_{(\Lambda,E)}(\mathfrak{e}) = ([\alpha, \beta]^{\mathfrak{g}^*}, 0),$$

for $(\hat{\alpha}, f), (\hat{\beta}, g) \in \Omega^1(G) \times C^\infty(G, \mathbb{R})$ satisfying $\hat{\alpha}(\mathfrak{e}) = \alpha, \hat{\beta}(\mathfrak{e}) = \beta$ and $f(\mathfrak{e}) = g(\mathfrak{e}) = 0$.

*Example 3.15:* Let $G$ be a connected simply connected abelian Lie group of dimension $n$ and $(\Lambda, E)$ be a Jacobi structure on $G$ such that $\Lambda$ is a multiplicative 2-vector and $E$ is a bi-invariant vector field. Then, $G$ is isomorphic, as a Lie group, to the dual space $\mathfrak{g}^*$ of a real vector space $\mathfrak{g}$ of dimension $n$, $\Lambda$ is a linear 2-vector on $\mathfrak{g}^*$ and there exists $\varphi \in \mathfrak{g}^*$ satisfying that $E = -C_\varphi$, $C_\varphi$ being the constant vector field on $\mathfrak{g}^*$ induced by $\varphi$. Thus, from (2.2), one can deduce that the Jacobi bracket of two linear functions on $\mathfrak{g}^*$ is again linear and that the Jacobi bracket of a linear function and the constant function 1 is a constant function. Therefore, using the results in [14] (see Theorem 2 and Example 1 in [14]) we conclude that $\mathfrak{g}$ is a Lie algebra with Lie bracket $[\, ,\,]^{\mathfrak{g}}$ and that

$$\Lambda = \Lambda_{\mathfrak{g}^*} + R \wedge C_\varphi, \quad E = -C_\varphi, \tag{3.20}$$

where $\Lambda_{\mathfrak{g}^*}$ is the Lie-Poisson structure on $\mathfrak{g}^*$, $R$ is the radial vector field on $\mathfrak{g}^*$ and $\varphi \in \mathfrak{g}^*$ is a 1-cocycle of $(\mathfrak{g}, [\, ,\,]^{\mathfrak{g}})$. The generalized Lie bialgebra associated with the Jacobi structure $(\Lambda, E)$ on $\mathfrak{g}^*$ is $((\mathfrak{g}^*, 0), (\mathfrak{g}, \varphi))$ and the Lie bracket on $\mathfrak{g}^*$ is trivial.

Conversely, if $(\mathfrak{g}, [\, ,\,]^{\mathfrak{g}})$ is a real Lie algebra of dimension $n$, $\varphi \in \mathfrak{g}^*$ is a 1-cocycle of $(\mathfrak{g}, [\, ,\,]^{\mathfrak{g}})$ and $(\Lambda, E)$ is the pair given by (3.20) then $\mathfrak{g}^*$ is a connected simply connected abelian Lie group and $(\Lambda, E)$ is a Jacobi structure on $\mathfrak{g}^*$ (see Theorem 1 in [14]). Moreover, it is clear that $\Lambda$ is multiplicative (linear) and that $E$ is bi-invariant (constant).

## 4. Coboundary generalized Lie bialgebras

From (3.1) and (3.7) we deduce that if $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, X_0))$ is a generalized Lie bialgebra then $d_{*X_0}$ is a 1-cocycle on $\mathfrak{g}$ with respect to the representation $ad_{(\phi_0,1)}\colon \mathfrak{g} \times \wedge^2\mathfrak{g} \to \wedge^2\mathfrak{g}$. In this Section, we will propose a method to obtain generalized Lie bialgebras such that $d_{*X_0}$ is a 1-coboundary (i.e., there exists $r \in \wedge^2\mathfrak{g}$ satisfying that $d_{*X_0}X = ad_{(\phi_0,1)}(X)(r)$, for $X \in \mathfrak{g}$). It is a generalization of the well-known Yang–Baxter equation method to obtain Lie bialgebras (see, for instance, [37]). It is clear that our method will allow us to obtain connected Lie groups such that their corresponding Lie algebras are generalized Lie bialgebras.

THEOREM 4.1: *Let $(\mathfrak{g}, [,]^{\mathfrak{g}})$ be a real Lie algebra of finite dimension. Suppose that $\phi_0 \in \mathfrak{g}^*$ is a 1-cocycle and that $r \in \wedge^2\mathfrak{g}$ and $X_0 \in \mathfrak{g}$ are such that*

$$[r,r]^{\mathfrak{g}} - 2X_0 \wedge r \text{ is } ad_{(\phi_0,1)}\text{-invariant}, \quad [X_0, r]^{\mathfrak{g}} = 0,$$
$$i(\phi_0)(r) - X_0 \text{ is } ad_{(\phi_0,0)}\text{-invariant}.$$

*If $[,]^{\mathfrak{g}^*}$ is the bracket on $\mathfrak{g}^*$ given by*

$$[\alpha,\beta]^{\mathfrak{g}^*} = coad_{\#_r(\beta)}\alpha - coad_{\#_r(\alpha)}\beta + r(\alpha,\beta)\phi_0 + i(X_0)(\alpha\wedge\beta), \tag{4.1}$$

*for $\alpha, \beta \in \mathfrak{g}^*$, where coad: $\mathfrak{g}\times\mathfrak{g}^* \to \mathfrak{g}^*$ is the coadjoint representation of $\mathfrak{g}$ over $\mathfrak{g}^*$, that is, $(coad(X)(\alpha))(Y) = -\alpha[X,Y]^{\mathfrak{g}}$, for $X, Y \in \mathfrak{g}$, then $(\mathfrak{g}^*, [,]^{\mathfrak{g}^*})$ is a Lie algebra and the pair $((\mathfrak{g},\phi_0),(\mathfrak{g}^*, X_0))$ is a generalized Lie bialgebra.*

*Proof:* Let $G$ be a connected simply connected Lie group with Lie algebra $\mathfrak{g}$.

We define a 2-vector $\Lambda$ and a vector field $E$ on $G$ by

$$\Lambda = \tilde{r} - e^{-\sigma}\bar{r}, \qquad E = -\tilde{X}_0, \tag{4.2}$$

where $\sigma$ is the unique multiplicative function satisfying that $(\delta\sigma)(\mathfrak{e}) = \phi_0$.

From Example 3.10, we have that $\Lambda$ is a $\sigma$-multiplicative 2-vector. On the other hand,

$$\#_\Lambda(\delta\sigma) - \tilde{X}_0 + e^{-\sigma}\bar{X}_0 = \big(\widetilde{i(\phi_0)(r) - X_0}\big) - e^{-\sigma}\big(\overline{i(\phi_0)(r) - X_0}\big).$$

Therefore, since $i(\phi_0)(r) - X_0$ is $ad_{(\phi_0,0)}$-invariant, we obtain that

$$\#_\Lambda(\delta\sigma) = \tilde{X}_0 - e^{-\sigma}\bar{X}_0 \tag{4.3}$$

(see Example 3.10). Note that, from this equality, we deduce that $\tilde{X}_0(\sigma) = \bar{X}_0(\sigma) = \phi_0(X_0) = 0$ (see the proof of Theorem 3.13).

On the other hand, using (4.2), (4.3) and the properties of the Schouten–Nijenhuis bracket, it follows that

$$[\Lambda,\Lambda] - 2E\wedge\Lambda = -\Big\{\big(\widetilde{[r,r]^{\mathfrak{g}} - 2X_0\wedge r}\big) - e^{-2\sigma}\big(\overline{[r,r]^{\mathfrak{g}} - 2X_0\wedge r}\big)\Big\}.$$

Thus, since $[r,r]^{\mathfrak{g}} - 2X_0\wedge r$ is $ad_{(\phi_0,1)}$-invariant, $[\Lambda,\Lambda] = 2E\wedge\Lambda$. Moreover,

$$\mathcal{L}_E\Lambda = -\mathcal{L}_{\tilde{X}_0}\tilde{r} - e^{-\sigma}\tilde{X}_0(\sigma)\bar{r} + e^{-\sigma}\mathcal{L}_{\tilde{X}_0}\bar{r} = 0.$$

Consequently, the pair $(\Lambda, E)$ is a Jacobi structure. Furthermore, from (3.8) and (4.2), we deduce that the intrinsic derivative of $\Lambda$ at $\mathfrak{e}$ is given by

$$(\delta_{\mathfrak{e}}\Lambda)(X) = -ad_{(\phi_0,1)}(X)(r) = -[X,r]^{\mathfrak{g}} + \phi_0(X)r, \tag{4.4}$$

for $X \in \mathfrak{g}$. Using this fact and Theorem 3.13, we conclude that the bracket on $\mathfrak{g}^*$ given by (4.1) is a Lie bracket and that the pair $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, X_0))$ is a generalized Lie bialgebra. ∎

*Remark 4.2:*

(i) Since $d_{*X_0}X = -(\delta_{\mathfrak{e}}\Lambda)(X)$, for all $X \in \mathfrak{g}$, we obtain that (see (4.4)) $d_* s = [s,r]^{\mathfrak{g}} - 2X_0 \wedge s - i(\phi_0)(s) \wedge r$, for all $s \in \wedge^2\mathfrak{g}$. In particular,

$$d_* r = [r,r]^{\mathfrak{g}} - 2X_0 \wedge r - i(\phi_0)r \wedge r. \tag{4.5}$$

(ii) If $X \in \mathfrak{g}$, it follows that (see (4.1))

$$\begin{aligned}[\alpha,\beta]^{\mathfrak{g}^*}(X) = &- [X,r]^{\mathfrak{g}}(\alpha,\beta) + r(\alpha,\beta)\phi_0(X) \\ &+ \alpha(X_0)\beta(X) - \beta(X_0)\alpha(X).\end{aligned} \tag{4.6}$$

Now, using Theorem 4.1, we have

COROLLARY 4.3: *Let $(\mathfrak{g}, [,]^{\mathfrak{g}})$ be a real Lie algebra of finite dimension. Suppose that $\phi_0 \in \mathfrak{g}^*$ is a 1-cocycle and that $r \in \wedge^2\mathfrak{g}$ and $X_0 \in \mathfrak{g}$ are such that $i(\phi_0)(r) = X_0$ and $(r, X_0)$ is an algebraic Jacobi structure on $\mathfrak{g}$ (to be defined in Appendix A). If $[,]^{\mathfrak{g}^*}$ is the Lie bracket on $\mathfrak{g}^*$ given by (4.1), then $(\mathfrak{g}^*, [,]^{\mathfrak{g}^*})$ is a Lie algebra and the pair $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, X_0))$ is a generalized Lie bialgebra. Moreover, the linear map $-\#_r\colon \mathfrak{g}^* \to \mathfrak{g}$ is a Lie algebra homomorphism.*

*Proof:* From Theorem 4.1 and Definition A.1 (see Appendix A), we deduce that the pair $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, X_0))$ is a generalized Lie bialgebra. On the other hand, if $\alpha, \beta, \gamma \in \mathfrak{g}^*$ then the equality $[r,r]^{\mathfrak{g}}(\alpha,\beta,\gamma) = 2(X_0 \wedge r)(\alpha,\beta,\gamma)$ implies that $\gamma[\#_r(\alpha), \#_r(\beta)]^{\mathfrak{g}} = [\alpha,\beta]^{\mathfrak{g}^*}(\#_r(\gamma))$ and therefore

$$\#_r([\alpha,\beta]^{\mathfrak{g}^*}) = -[\#_r(\alpha), \#_r(\beta)]^{\mathfrak{g}}. \quad ∎$$

*Remark 4.4:* Let $(\mathfrak{g}, [,]^{\mathfrak{g}})$ be a real Lie algebra of finite dimension. Assume that $(\Omega, \omega)$ is an algebraic locally conformal symplectic (l.c.s.) structure on $\mathfrak{g}$ and denote by $(r, X_0)$ the corresponding algebraic Jacobi structure on $\mathfrak{g}$ (see Appendix A). Then, using Corollary 4.3 and the fact that $X_0 = -\#_r(\omega)$, we deduce that the pair $((\mathfrak{g}, -\omega), (\mathfrak{g}^*, X_0))$ is a generalized Lie bialgebra. Furthermore, since $\#_r\colon \mathfrak{g}^* \to \mathfrak{g}$ is a linear isomorphism (see Appendix A), it follows that $\mathfrak{g}^*$ is isomorphic, as a Lie algebra, to $\mathfrak{g}$.

## 5. Examples of generalized Lie bialgebras

First, we will give some examples of generalized Lie bialgebras which are obtained using Theorem 4.1 and Corollary 4.3.

5.1. Generalized Lie bialgebras from contact Lie algebras. Let $(\mathfrak{g}, [,]^{\mathfrak{g}})$ be a Lie algebra endowed with an algebraic contact 1-form $\eta$ and let $X_0$ be the Reeb vector of $\mathfrak{g}$ (see Appendix A). If $\mathcal{Z}(\mathfrak{g})$ is the center of $\mathfrak{g}$ and $X \in \mathcal{Z}(\mathfrak{g})$ then it is clear that $i(X)(d\eta) = 0$. This implies that $X \in < X_0 >$. Thus, $\mathcal{Z}(\mathfrak{g}) \subseteq < X_0 >$ (see [7]). Therefore, we have two possibilities: $\mathcal{Z}(\mathfrak{g}) = \{0\}$ or $\mathcal{Z}(\mathfrak{g}) = < X_0 >$.

If $\mathcal{Z}(\mathfrak{g}) = < X_0 >$ then Diatta [7] proved that $\mathfrak{g}$ is the central extension of a symplectic Lie algebra $(\mathfrak{h}, [,]^{\mathfrak{h}})$ by $\mathbb{R}$ via the 2-cocycle $\Omega$, $\Omega$ being the algebraic symplectic structure on $\mathfrak{h}$. Conversely, if $(\mathfrak{h}, [,]^{\mathfrak{h}})$ is a symplectic Lie algebra, with algebraic symplectic 2-form $\Omega$, and on the direct product $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}$ we consider the Lie bracket $[,]^{\mathfrak{g}}$ given by

$$[(X,\lambda),(Y,\mu)]^{\mathfrak{g}} = ([X,Y]^{\mathfrak{h}}, -\Omega(X,Y)), \text{ for } (X,\lambda),(Y,\mu) \in \mathfrak{g}, \tag{5.1}$$

then $\eta = (0,1) \in \mathfrak{h}^* \oplus \mathbb{R} \cong \mathfrak{g}^*$ is an algebraic contact 1-form on $\mathfrak{g}$. Moreover, since $X_0 = (0,1) \in \mathfrak{h} \oplus \mathbb{R} = \mathfrak{g}$, we deduce that $\mathcal{Z}(\mathfrak{g}) = < X_0 >$ (see [7]).

Now, suppose that $r$ is the algebraic Poisson 2-vector on $\mathfrak{h}$ associated with the algebraic symplectic structure $\Omega$. Then, the pair $(r, X_0)$ is the algebraic Jacobi structure on $\mathfrak{g}$ associated with the contact 1-form $\eta$ (see (A.1), (A.2), (A.5) and (A.6) in Appendix A). Thus, using Theorem 4.1 and the fact that $X_0 \in \mathcal{Z}(\mathfrak{g})$, we can define a Lie bracket $[,]^{\mathfrak{g}^*}$ on $\mathfrak{g}^*$ in such a way that the pair $((\mathfrak{g}, 0), (\mathfrak{g}^*, X_0))$ is a generalized Lie bialgebra.

On the other hand, from Corollary 4.3 and since $r$ is a solution of the classical Yang–Baxter equation on $\mathfrak{h}$, it follows that there exists a Lie bracket $[,]^{\mathfrak{h}^*}$ on $\mathfrak{h}^*$ in such a way that the pair $(\mathfrak{h}, \mathfrak{h}^*)$ is a Lie bialgebra. In fact, the Lie algebras $(\mathfrak{h}, [,]^{\mathfrak{h}})$ and $(\mathfrak{h}^*, [,]^{\mathfrak{h}^*})$ are isomorphic and, using (4.1), we get that $[(\alpha,\lambda),(\beta,\mu)]^{\mathfrak{g}^*} = ([\alpha,\beta]^{\mathfrak{h}^*}, 0)$, for $(\alpha,\lambda),(\beta,\mu) \in \mathfrak{h}^* \oplus \mathbb{R} \cong \mathfrak{g}^*$. Consequently, $\mathfrak{g}^*$ is isomorphic, as a Lie algebra, to the direct product $\mathfrak{h} \oplus \mathbb{R}$.

We illustrate the preceding construction with a simple example.

Let $(\mathfrak{h}, [,]^{\mathfrak{h}})$ be the abelian Lie algebra of dimension $2n$ and $\Omega$ the usual symplectic 2-form. Then, $\mathfrak{h} \oplus \mathbb{R}$ endowed with the Lie bracket given by (5.1) is just the Lie algebra $\mathfrak{h}(1,n)$ of the generalized Heisenberg group $H(1,n)$ (see [13]) and the 1-form $\eta$ is just the usual algebraic contact 1-form on $\mathfrak{h}(1,n)$. In this case, the Lie algebra $\mathfrak{h}(1,n)^*$ is abelian.

*Remark 5.1:* A complete description of symplectic Lie algebras of dimension 4 was obtained in [32] (for a detailed study of symplectic Lie algebras, see also [5, 26]). Thus, one can determine all contact Lie algebras of dimension 5 with center of dimension 1 and from there, using Theorem 4.1, obtain different examples of generalized Lie bialgebras.

Now, we will give two examples of generalized Lie bialgebras $((\mathfrak{g},\phi_0),(\mathfrak{g}^*,X_0))$ associated to an algebraic contact structure on $\mathfrak{g}$ but in both cases $\phi_0 \neq 0$. In the first example, $X_0 \in \mathcal{Z}(\mathfrak{g})$. However, $X_0 \notin \mathcal{Z}(\mathfrak{g})$ in the second one.

**1.-** Let $(\mathfrak{h},[\,,]^{\mathfrak{h}})$ be the nonabelian solvable Lie algebra of dimension 2. We can find a basis $\{e_1,e_2\}$ of $\mathfrak{h}$ such that $[e_1,e_2]^{\mathfrak{h}}=e_1$. If we consider on $\mathfrak{g}=\mathfrak{h}\oplus\mathbb{R}$ the Lie bracket given by (5.1), it is easy to prove that $\phi_0=-e^2$ is a 1-cocycle of $\mathfrak{g}$, $\{e^1,e^2\}$ being the dual basis of $\{e_1,e_2\}$. We also have that $\eta=(0,1)\in\mathfrak{h}^*\oplus\mathbb{R}\cong\mathfrak{g}^*$ is an algebraic contact 1-form on $\mathfrak{g}$ and that $(r,X_0)$ is the corresponding Jacobi structure, where $r=e_2\wedge e_1$ and $X_0=(0,1)\in\mathfrak{h}\oplus\mathbb{R}=\mathfrak{g}$. On the other hand, using (5.1), we deduce that $i(\phi_0)r-X_0$ is $ad^{\mathfrak{g}}_{(\phi_0,0)}$-invariant. Thus, from Theorem 4.1, $((\mathfrak{g},\phi_0),(\mathfrak{g}^*,X_0))$ is a generalized Lie bialgebra. Note that the Lie algebra $(\mathfrak{g},[\,,]^{\mathfrak{g}})$ is isomorphic to the direct product $\mathfrak{h}\oplus\mathbb{R}$ and that $\mathfrak{g}^*$ is the abelian Lie algebra of dimension 3 (see (4.1)).

**2.-** Let $(\mathfrak{g},[\,,]^{\mathfrak{g}})$ be the solvable Lie algebra of dimension 3 with basis $\{e_1,e_2,e_3\}$ such that

$$[e_1,e_2]^{\mathfrak{g}}=0,\quad [e_1,e_3]^{\mathfrak{g}}=e_1,\quad [e_3,e_2]^{\mathfrak{g}}=e_2.$$

Take $r=e_3\wedge(e_1-e_2)$ and $X_0=e_1+e_2$. It is easy to prove that $(r,X_0)$ is an algebraic Jacobi structure on $\mathfrak{g}$ which is associated to an algebraic contact structure. Moreover, if $\{e^1,e^2,e^3\}$ is the dual basis of $\mathfrak{g}^*$ then $\phi_0==e^3$ is a 1-cocycle of $\mathfrak{g}$ and $i(\phi_0)r-X_0$ is $ad^{\mathfrak{g}}_{(\phi_0,0)}$-invariant. Therefore, from Theorem 4.1, we deduce that $((\mathfrak{g},\phi_0),(\mathfrak{g}^*,X_0))$ is a generalized Lie bialgebra. The Lie bracket on $\mathfrak{g}^*$ is characterized by

$$[e^1,e^2]^{\mathfrak{g}^*}=0,\quad [e^3,e^1]^{\mathfrak{g}^*}=e^3,\quad [e^2,e^3]^{\mathfrak{g}^*}=e^3.$$

5.2. Generalized Lie bialgebras from locally conformal symplectic Lie algebras. Suppose that $(r_{\mathfrak{h}},X_0)$ is an algebraic contact structure on a Lie algebra $(\mathfrak{h},[\,,]^{\mathfrak{h}})$. If we consider on the direct product of Lie algebras $\mathfrak{g}=\mathfrak{h}\oplus\mathbb{R}$ the 2-vector

$$r=r_{\mathfrak{h}}+e_0\wedge X_0, \tag{5.2}$$

where $e_0 = (0,1) \in \mathfrak{h} \oplus \mathbb{R} = \mathfrak{g}$, then $(r, X_0)$ is an algebraic l.c.s. structure and, using Remark 4.4, $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, X_0))$ is a generalized Lie bialgebra, with $\phi_0 = (0,1) \in \mathfrak{h}^* \oplus \mathbb{R} \cong \mathfrak{g}^*$. In addition, the Lie algebras $\mathfrak{g}$ and $\mathfrak{g}^*$ are isomorphic (see Remark 4.4).

*Remark 5.2:* If $H$ is a connected Lie group with Lie algebra $\mathfrak{h}$ then the pair $(\bar{r}, \bar{X}_0)$ defines, on the direct product $G = H \times \mathbb{R}$, a left invariant l.c.s. structure of the first kind in the sense of Vaisman [36].

In the case when $\mathcal{Z}(\mathfrak{h}) =< X_0 >$ we have that the pair $((\mathfrak{h}, 0), (\mathfrak{h}^*, X_0))$ is a generalized Lie bialgebra (see Section 5.1). Moreover, from (4.1) and (5.2), we deduce that the Lie bracket $[,]^{\mathfrak{g}^*}$ on $\mathfrak{g}^*$ can be described, in terms of the Lie bracket $[,]^{\mathfrak{h}^*}$ of $\mathfrak{h}^*$, as follows:

$$[(\alpha, \lambda), (\beta, \mu)]^{\mathfrak{g}^*} = ([\alpha, \beta]^{\mathfrak{h}^*}, r_{\mathfrak{h}}(\alpha, \beta)),$$

for $(\alpha, \lambda), (\beta, \mu) \in \mathfrak{h}^* \oplus \mathbb{R} \cong \mathfrak{g}^*$. Thus, since $r_{\mathfrak{h}}$ is a 2-cocycle of the Lie algebra $(\mathfrak{h}^*, [,]^{\mathfrak{h}^*})$ (see (4.5)), it follows that $\mathfrak{g}^*$ is the central extension of $\mathfrak{h}^*$ by $\mathbb{R}$ via the 2-cocycle $r_{\mathfrak{h}}$.

On the other hand, in [7], Diatta proved that if $(\mathfrak{h}', [,]^{\mathfrak{h}'})$ is an exact symplectic Lie algebra then one can define on the direct product $\mathfrak{h} = \mathfrak{h}' \oplus \mathbb{R}$ a Lie bracket in such a way that $\mathfrak{h}$ is a contact Lie algebra, with trivial center, and $\mathfrak{h}'$ is a Lie subalgebra of $\mathfrak{h}$. Using this construction we can also obtain different examples of generalized Lie bialgebras. Next, we will show an explicit example.

Let $\mathfrak{sl}(2,\mathbb{R})$ be the Lie algebra of the special linear group $SL(2,\mathbb{R})$. Then, there exists a basis $\{e_1, e_2, e_3\}$ of $\mathfrak{sl}(2,\mathbb{R})$ such that

$$[e_1, e_2]^{\mathfrak{sl}(2,\mathbb{R})} = 2e_2, \quad [e_3, e_1]^{\mathfrak{sl}(2,\mathbb{R})} = 2e_3, \quad [e_2, e_3]^{\mathfrak{sl}(2,\mathbb{R})} = e_1.$$

It is clear that $\mathfrak{sl}(2,\mathbb{R})$ admits exact symplectic Lie subalgebras and, therefore, we can apply Diatta's method in order to obtain algebraic contact structures on $\mathfrak{sl}(2,\mathbb{R})$. In fact, if $\lambda^1, \lambda^2$ and $\lambda^3$ are real numbers satisfying the relation $(\lambda^1)^2 + 4\lambda^2\lambda^3 \neq 0$ then the pair $(r_{\mathfrak{sl}(2,\mathbb{R})}, X_0)$ given by

$$r_{\mathfrak{sl}(2,\mathbb{R})} = \lambda^1 e_2 \wedge e_3 + \lambda^2 e_1 \wedge e_2 + \lambda^3 e_3 \wedge e_1, \quad X_0 = -(\lambda^1 e_1 + 2\lambda^2 e_2 + 2\lambda^3 e_3),$$

defines an algebraic Jacobi structure on $\mathfrak{sl}(2,\mathbb{R})$ which is associated to an algebraic contact structure. Consequently, since $\mathfrak{gl}(2,\mathbb{R})$ (the Lie algebra of the general linear group $GL(2,\mathbb{R})$) is isomorphic to the direct product $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}$, we conclude that the pair $((\mathfrak{gl}(2,\mathbb{R}), \phi_0), (\mathfrak{gl}(2,\mathbb{R})^*, X_0))$ is a generalized Lie bialgebra, where $\phi_0 = (0,1) \in \mathfrak{sl}(2,\mathbb{R})^* \oplus \mathbb{R} \cong \mathfrak{gl}(2,\mathbb{R})^*$.

Finally, we remark that there exist examples of contact Lie algebras with trivial center which do not admit symplectic Lie subalgebras. An interesting case is $\mathfrak{su}(2)$, the Lie algebra of the special unitary group $SU(2)$. We can consider a basis $\{e_1, e_2, e_3\}$ of $\mathfrak{su}(2)$ such that

$$[e_1, e_2]^{\mathfrak{su}(2)} = e_3, \quad [e_3, e_1]^{\mathfrak{su}(2)} = e_2, \quad [e_2, e_3]^{\mathfrak{su}(2)} = e_1.$$

Then, if $\lambda^1, \lambda^2$ and $\lambda^3$ are real numbers, $(\lambda^1, \lambda^2, \lambda^3) \neq (0,0,0)$, we have that the pair $(r_{\mathfrak{su}(2)}, X_0)$ given by

$$r_{\mathfrak{su}(2)} = \lambda^1 e_2 \wedge e_3 + \lambda^2 e_3 \wedge e_1 + \lambda^3 e_1 \wedge e_2, \quad X_0 = -(\lambda^1 e_1 + \lambda^2 e_2 + \lambda^3 e_3),$$

defines an algebraic Jacobi structure on $\mathfrak{su}(2)$ which is associated to an algebraic contact structure. Thus, since $\mathfrak{u}(2)$ (the Lie algebra of the unitary group $U(2)$) is isomorphic to the direct product $\mathfrak{su}(2) \oplus \mathbb{R}$, we deduce that the pair $((\mathfrak{u}(2), \phi_0), (\mathfrak{u}(2)^*, X_0))$ is a generalized Lie bialgebra, where $\phi_0 = (0,1) \in \mathfrak{su}(2)^* \oplus \mathbb{R} \cong \mathfrak{u}(2)^*$.

We will treat again this example in Section 6.

5.3. Other examples of generalized Lie bialgebras. All the examples of generalized Lie bialgebras $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, X_0))$ considered in Sections 5.1 and 5.2 have been obtained from an algebraic Jacobi structure $(r, X_0)$ on $\mathfrak{g}$. However, the hypotheses of Theorem 4.1 do not necessarily imply that the pair $(r, X_0)$ is an algebraic Jacobi structure on $\mathfrak{g}$, as is shown in the following simple example.

Let $\mathfrak{h}$ be the abelian Lie algebra of dimension 3. Take $\{e_1, e_2, e_3\}$ a basis of $\mathfrak{h}$ and let $\{e^1, e^2, e^3\}$ be the dual basis of $\mathfrak{h}^*$. Denote by $\Psi$ the endomorphism of $\mathfrak{h}$ given by $\Psi = \frac{1}{2} e_1 \otimes e^1 + \frac{1}{2} e_2 \otimes e^2 + e_3 \otimes e^3$. $\Psi$ is a 1-cocycle with respect to the adjoint representation of $\mathfrak{h}$. Thus, we can consider the representation of $\mathbb{R}$ on $\mathfrak{h}$ given by $\mathbb{R} \times \mathfrak{h} \to \mathfrak{h}$, $(\lambda, X) \mapsto \lambda \Psi(X)$, and the corresponding semi-direct product $\mathfrak{g} = \mathfrak{h} \times_\Psi \mathbb{R}$. We can choose a basis $\{e_1, e_2, e_3, e_4\}$ of $\mathfrak{g}$ such that

$$[e_4, e_1]^{\mathfrak{g}} = \frac{1}{2} e_1, \quad [e_4, e_2]^{\mathfrak{g}} = \frac{1}{2} e_2, \quad [e_4, e_3]^{\mathfrak{g}} = e_3,$$

and the other brackets are zero. Suppose that $\{e^1, e^2, e^3, e^4\}$ is the dual basis of $\mathfrak{g}^*$. If $r \in \wedge^2 \mathfrak{g}$, $X_0 \in \mathfrak{g}$ and $\phi_0 \in \mathfrak{g}^*$ are defined by

$$r = e_1 \wedge e_2 - 2 e_3 \wedge e_4, \quad X_0 = e_3, \quad \phi_0 = e^4,$$

then $r$, $X_0$ and $\phi_0$ satisfy the hypotheses of Theorem 4.1. However,

$$[r, r]^{\mathfrak{g}} - 2 X_0 \wedge r = 2 e_1 \wedge e_2 \wedge e_3 \neq 0 \quad \text{and} \quad i(\phi_0) r - X_0 = e_3 \neq 0.$$

Moreover, a direct computation shows that

$$[e^3, e^4]^{\mathfrak{g}^*} = e^4, \quad [e^i, e^j]^{\mathfrak{g}^*} = 0,$$

for $1 \leq i < j \leq 4$, $(i,j) \neq (3,4)$.

Finally, we will exhibit an example of a generalized Lie bialgebra $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, X_0))$ such that $\phi_0 \neq 0$ and $d_{*X_0}$ is not a 1-coboundary with respect to the representation $ad_{(\phi_0,1)}\colon \mathfrak{g} \times \wedge^2\mathfrak{g} \to \wedge^2\mathfrak{g}$. Note that in Section 3.2 (see Example 3.15), we obtained an example which satisfies this last condition but, in that case, $\phi_0 = 0$. On the other hand, all the examples of generalized Lie bialgebras that we have given in Section 5 are such that $d_{*X_0}$ is a 1-coboundary.

Let $\mathfrak{g}$ be the Lie algebra of dimension 4 with basis $\{e_1, e_2, e_3, e_4\}$ satisfying

$$[e_4, e_1]^{\mathfrak{g}} = e_1, \quad [e_4, e_2]^{\mathfrak{g}} = e_2, \quad [e_4, e_3]^{\mathfrak{g}} = e_3$$

and the other brackets being zero. If $\{e^1, e^2, e^3, e^4\}$ is the dual basis of $\mathfrak{g}^*$, we consider on $\mathfrak{g}^*$ the Lie bracket $[,]^{\mathfrak{g}^*}$ characterized by

$$[e^1, e^2]^{\mathfrak{g}^*} = e^3, \quad [e^1, e^4]^{\mathfrak{g}^*} = e^4, \quad [e^i, e^j]^{\mathfrak{g}^*} = 0,$$

for $1 \leq i < j \leq 4$, $(i,j) \neq (1,2), (1,4)$. Then, the pair $((\mathfrak{g}, e^4), (\mathfrak{g}^*, e_1))$ is a generalized Lie bialgebra. Moreover, it is easy to prove that there does not exist $r \in \wedge^2\mathfrak{g}$ such that $d_{*X_0}X = ad_{(\phi_0,1)}(X)(r)$, for all $X \in \mathfrak{g}$.

## 6. Compact generalized Lie bialgebras

Several authors have devoted special attention to the study of compact Lie bialgebras and an important result in this direction is the following one [28] (see also [31]): every connected compact semisimple Lie group has a nontrivial Poisson Lie group structure.

In this Section, we will describe the structure of a generalized Lie bialgebra $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, X_0))$, $\mathfrak{g}$ being a compact Lie algebra (that is, $\mathfrak{g}$ is the Lie algebra of a compact connected Lie group).

If $\phi_0 = 0$ and $X_0 = 0$, the pair $(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie bialgebra. Thus, we will suppose that $\phi_0 \neq 0$ or $X_0 \neq 0$. Note that if $\phi_0 = 0$ then $X_0 \in \mathcal{Z}(\mathfrak{g})$ (see (3.3)). On the other hand, if $\phi_0 \neq 0$ then we can consider an $ad$-invariant scalar product $<,> : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ and the vector $\bar{Y}_0 \in \mathfrak{g}$ characterized by the relation $\phi_0(X) = < X, \bar{Y}_0 >$, for $X \in \mathfrak{g}$. It is clear that $\bar{Y}_0 \neq 0$ and, moreover, using that $\phi_0$ is a 1-cocycle and the fact that $<,>$ is an $ad$-invariant scalar product, we obtain that $\bar{Y}_0 \in \mathcal{Z}(\mathfrak{g})$ (we remark that $\phi_0(Y_0) = 1$ with $Y_0 = \frac{\bar{Y}_0}{\phi_0(\bar{Y}_0)} \in \mathcal{Z}(\mathfrak{g})$).

Therefore, if $\phi_0 \neq 0$ or $X_0 \neq 0$, we have that $dim\mathcal{Z}(\mathfrak{g}) \geq 1$. This implies that a compact connected Lie group $G$ with Lie algebra $\mathfrak{g}$ cannot be semisimple.

Next, we will distinguish two cases:

*(a) The case $\phi_0 \neq 0$*

Let $\mathfrak{g}$ be a compact Lie algebra and $\phi_0 \in \mathfrak{g}^*$ a 1-cocycle, $\phi_0 \neq 0$. If $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$ and $(r, X_0)$ is an algebraic l.c.s. structure on $\mathfrak{h}$ such that $i(\phi_0)(r) = X_0$ then, from Corollary 4.3, we deduce that the pair $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, X_0))$ is a generalized Lie bialgebra, where the Lie bracket on $\mathfrak{g}^*$ is given by (4.1).

Using the above construction, we can obtain some examples of generalized Lie bialgebras $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, X_0))$, with $\phi_0 \neq 0$ and $\mathfrak{g}$ a compact Lie algebra.

*Examples 6.1:* (i) *Compact generalized Lie bialgebras of the first kind.* Let $\mathfrak{g}$ be a compact Lie algebra and $\mathfrak{h}$ an abelian Lie subalgebra of even dimension. Furthermore, assume that $r \in \wedge^2\mathfrak{h}$ is a nondegenerate 2-vector on $\mathfrak{h}$ (that is, $r$ comes from an algebraic symplectic structure on $\mathfrak{h}$) and that $\phi_0 \in \mathfrak{g}^*$ is a 1-cocycle on $\mathfrak{g}$ such that $\phi_0 \neq 0$ and $\phi_0 \in \mathfrak{h}^\circ$, $\mathfrak{h}^\circ$ being the annihilator of $\mathfrak{h}$. Then, $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, 0))$ is a generalized Lie bialgebra. The pair $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, 0))$ is said to be a compact generalized Lie bialgebra of the first kind.

(ii) *Compact generalized Lie bialgebras of the second kind.* Let $(\mathfrak{g}, [\,,]^{\mathfrak{g}})$ be a compact real Lie algebra. Suppose that $e_1, e_2 \in \mathfrak{g}$ are linearly independent and that $[e_1, e_2]^{\mathfrak{g}} = 0$. We consider the 2-vector $r$ and the vector $X_0$ on $\mathfrak{g}$ defined by $r = \lambda e_1 \wedge e_2$ and $X_0 = \lambda^1 e_1 + \lambda^2 e_2$, with $\lambda \in \mathbb{R} - \{0\}$ and $(\lambda^1, \lambda^2) \in \mathbb{R}^2 - \{(0,0)\}$. It is clear that $(r, X_0)$ is an algebraic Jacobi structure on $\mathfrak{g}$ which comes from an algebraic l.c.s. structure on the Lie subalgebra $\mathfrak{h} =< e_1, e_2 >$. Therefore, if $\phi_0 \in \mathfrak{g}^*$ is a 1-cocycle of $\mathfrak{g}$ such that $i(\phi_0)(r) = X_0$ (that is, $\phi_0(e_1) = \lambda^2/\lambda$ and $\phi_0(e_2) = -\lambda^1/\lambda$) then $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, X_0))$ is a generalized Lie bialgebra. The pair $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, X_0))$ is said to be a compact generalized Lie bialgebra of the second kind.

(iii) *Compact generalized Lie bialgebras of the third kind.* Let $(\mathfrak{g}, [\,,]^{\mathfrak{g}})$ be a nonabelian compact real Lie algebra. By the root space decomposition theorem, we know that there exist $e_1, e_2, e_3 \in \mathfrak{g}$ satisfying

$$[e_1, e_2]^{\mathfrak{g}} = e_3, \quad [e_3, e_1]^{\mathfrak{g}} = e_2, \quad [e_2, e_3]^{\mathfrak{g}} = e_1. \tag{6.1}$$

Now, suppose that $\phi_0 \in \mathfrak{g}^*$ is a 1-cocycle on $\mathfrak{g}$ and that $e_4$ is a vector of $\mathfrak{g}$ such that $\phi_0(e_4) = 1$, and $[e_4, e_i]^{\mathfrak{g}} = 0$, for $i = 1, 2, 3$ (note that if $\mathcal{Z}(\mathfrak{g}) \neq \{0\}$, then the existence of $\phi_0$ and $e_4$ is guaranteed). Then, we consider the 2-vector $r$ and the vector $X_0$ on $\mathfrak{g}$ defined by

$$r = \lambda^1(e_2 \wedge e_3 + e_1 \wedge e_4) + \lambda^2(e_3 \wedge e_1 + e_2 \wedge e_4) + \lambda^3(e_1 \wedge e_2 + e_3 \wedge e_4),$$

$$X_0 = -(\lambda^1 e_1 + \lambda^2 e_2 + \lambda^3 e_3),$$

with $(\lambda^1, \lambda^2, \lambda^3) \in \mathbb{R}^3 - \{(0,0,0)\}$. A direct computation proves that $(r, X_0)$ is an algebraic l.c.s. structure on the Lie subalgebra $\mathfrak{h} =< e_1, e_2, e_3, e_4 >$ (see Section 5.2). Moreover, $i(\phi_0)(r) = X_0$. Thus, $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, X_0))$ is a generalized Lie bialgebra. The pair $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, X_0))$ is said to be a compact generalized Lie bialgebra of the third kind.

Next, we will show that Examples 6.1 (i), (ii) and (iii) are the only examples of generalized Lie bialgebras $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, X_0))$, with $\phi_0 \neq 0$ and $\mathfrak{g}$ a compact Lie algebra.

THEOREM 6.2: *Let $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, X_0))$ be a generalized Lie bialgebra. Suppose that $\phi_0(Y_0) = 1$, with $Y_0 \in \mathcal{Z}(\mathfrak{g})$. Then, there exists a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a 2-vector $r \in \wedge^2\mathfrak{h} \subseteq \wedge^2\mathfrak{g}$ such that $X_0 \in \mathfrak{h}$ and:*

(i) *The pair $(r, X_0)$ defines an algebraic Jacobi structure on $\mathfrak{g}$ which is associated to an algebraic l.c.s. structure on $\mathfrak{h}$. Moreover, $i(\phi_0)(r) = X_0$.*

(ii) *The Lie bracket $[,]^{\mathfrak{g}^*}$ on $\mathfrak{g}^*$ is given by (4.1).*

*Proof:* Denote by $r$ the 2-vector on $\mathfrak{g}$ given by

$$r = -d_{*X_0} Y_0. \tag{6.2}$$

Using (3.2), (3.3), (6.2) and the fact that $Y_0 \in \mathcal{Z}(\mathfrak{g})$, we have that

$$i(\phi_0)(r) = X_0. \tag{6.3}$$

From (3.1), (6.2) and since $Y_0 \in \mathcal{Z}(\mathfrak{g})$, it follows that

$$0 = d_{*X_0}[X, Y_0]^{\mathfrak{g}} = -[X, r]^{\mathfrak{g}} + \phi_0(X) r + d_{*X_0} X, \tag{6.4}$$

for all $X \in \mathfrak{g}$. Therefore, using (3.2), (6.4) and that $X_0$ is a 1-cocycle on $(\mathfrak{g}^*, [,]^{\mathfrak{g}^*})$, we deduce that

$$[X_0, r]^{\mathfrak{g}} = 0. \tag{6.5}$$

On the other hand, using again (6.4) and the properties of the algebraic Schouten bracket $[,]^{\mathfrak{g}}$, we conclude that $[r', r]^{\mathfrak{g}} = d_* r' + 2X_0 \wedge r' + r \wedge i(\phi_0)(r')$, for $r' \in \wedge^2\mathfrak{g}$. Consequently (see (6.2) and (6.3)),

$$[r, r]^{\mathfrak{g}} - 2X_0 \wedge r = d_* r + r \wedge X_0 = d_{*X_0} r = 0. \tag{6.6}$$

Thus, the pair $(r, X_0)$ is an algebraic Jacobi structure on $\mathfrak{g}$ and the rank of $(r, X_0)$ is even (see (6.3), (6.5) and (6.6)). Therefore, using Proposition A.4

(see Appendix A), it follows that there exists a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $r \in \wedge^2\mathfrak{h}$, $X_0 \in \mathfrak{h}$ and the pair $(r, X_0)$ is associated to an algebraic l.c.s. structure on $\mathfrak{h}$.

Finally, from (4.6) and (6.4), we deduce that the Lie bracket on $\mathfrak{g}^*$ is given by (4.1). ∎

Now, we will describe the algebraic l.c.s. structures on a compact Lie algebra.

THEOREM 6.3: *Let $\mathfrak{h}$ be a compact Lie algebra of dimension $2k \geq 2$. Suppose that $(r, X_0)$ is an algebraic Jacobi structure on $\mathfrak{h}$ which is associated to an algebraic l.c.s. structure.*

(i) *If $X_0 = 0$ then $\mathfrak{h}$ is the abelian Lie algebra and $r$ is a nondegenerate 2-vector on $\mathfrak{h}$.*

(ii) *If $X_0 \neq 0$ and $k = 1$ then $\mathfrak{h}$ is the abelian Lie algebra and $r$ is an arbitrary 2-vector on $\mathfrak{h}$, $r \neq 0$.*

(iii) *If $X_0 \neq 0$ and $k \geq 2$ then $k = 2$, $\mathfrak{h}$ is isomorphic to $\mathfrak{u}(2)$ and*

$$r = \lambda^1(e_2 \wedge e_3 + e_1 \wedge e_4) + \lambda^2(e_3 \wedge e_1 + e_2 \wedge e_4) + \lambda^3(e_1 \wedge e_2 + e_3 \wedge e_4),$$
$$X_0 = -(\lambda^1 e_1 + \lambda^2 e_2 + \lambda^3 e_3),$$

*where $(\lambda^1, \lambda^2, \lambda^3) \in \mathbb{R}^3 - \{(0,0,0)\}$ and $\{e_1, e_2, e_3, e_4\}$ is a basis of $\mathfrak{h}$ such that $e_4 \in \mathcal{Z}(\mathfrak{h})$ and*

$$[e_1, e_2]^{\mathfrak{h}} = e_3, \quad [e_3, e_1]^{\mathfrak{h}} = e_2, \quad [e_2, e_3]^{\mathfrak{h}} = e_1. \tag{6.7}$$

*Proof:* Denote by $(\Omega, \omega)$ the algebraic l.c.s. structure on $\mathfrak{h}$ associated with the pair $(r, X_0)$.

(i) If $X_0 = 0$, we obtain that $\omega = 0$ and $\Omega$ is an algebraic symplectic structure on $\mathfrak{h}$ (see (A.4)). Thus, since $\mathfrak{h}$ is a compact Lie algebra, (i) follows using the results in [3] (see also [26]).

(ii) This is trivial.

(iii) Suppose that $X_0 \neq 0$ and that $k \geq 2$. Then, $\omega \neq 0$. Moreover, we can consider an $ad$-invariant scalar product $<,> : \mathfrak{h} \times \mathfrak{h} \to \mathbb{R}$ and the vector $\bar{Y}_0$ of $\mathfrak{h}$ characterized by the relation

$$\omega(X) = < X, \bar{Y}_0 >, \quad \text{for } X \in \mathfrak{h}. \tag{6.8}$$

Using (6.8) and the fact that $\omega$ is a 1-cocycle, we deduce that $\bar{Y}_0 \in \mathcal{Z}(\mathfrak{h})$. Consequently,

$$\omega(Y_0) = 1, \tag{6.9}$$

with $Y_0 = \frac{\tilde{Y}_0}{\omega(\tilde{Y}_0)} \in \mathcal{Z}(\mathfrak{h})$.

On the other hand, if $\mathfrak{h}' \subseteq \mathfrak{h}$ is the annihilator of the subspace generated by $\omega$, it is clear that $\mathfrak{h}'$ is a Lie subalgebra of $\mathfrak{h}$. In fact, using (6.9) and since $Y_0 \in \mathcal{Z}(\mathfrak{h})$ and $\omega$ is a 1-cocycle, it follows that $\mathfrak{h}$ is isomorphic, as a Lie algebra, to the direct product $\mathfrak{h}' \oplus \mathbb{R}$. In addition, we will show that $\mathfrak{h}'$ admits an algebraic contact structure. For this purpose, we define the 1-form $\bar{\eta}$ on $\mathfrak{h}$ given by

$$\bar{\eta} = -i(Y_0)(\Omega). \tag{6.10}$$

Using the equality $\omega = i(X_0)(\Omega)$, we have that

$$\bar{\eta}(X_0) = 1. \tag{6.11}$$

Moreover, from (6.9), (6.10), (A.4) and since $Y_0 \in \mathcal{Z}(\mathfrak{h})$, we deduce that

$$0 = \mathcal{L}_{Y_0}\Omega = i(Y_0)(d\Omega) + d(i(Y_0)(\Omega)) = \Omega + \omega \wedge \bar{\eta} - d\bar{\eta}. \tag{6.12}$$

In particular (see (6.9), (6.10) and (6.11))

$$i(X_0)(d\bar{\eta}) = i(Y_0)(d\bar{\eta}) = 0. \tag{6.13}$$

Thus, the condition $\Omega^k = \Omega \wedge \overset{(k}{\ldots} \wedge \Omega \neq 0$ implies that $\omega \wedge \bar{\eta} \wedge (d\bar{\eta})^{k-1} \neq 0$. Therefore, the restriction $\eta$ of $\bar{\eta}$ to $\mathfrak{h}'$ is an algebraic contact 1-form on $\mathfrak{h}'$. Furthermore, if $(r', X_0')$ is the algebraic Jacobi structure on $\mathfrak{h}'$ associated with the contact 1-form then, from relations (6.9)–(6.13) and the results in Appendix A, we obtain that $r' = r + Y_0 \wedge X_0$ and $X_0' = X_0$. Consequently, taking $e_4 = -Y_0$ and using Proposition B.1 (see Appendix B), we prove (iii). ∎

Now, suppose that $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, X_0))$ is a generalized Lie bialgebra, with $\phi_0 \neq 0$ and $\mathfrak{g}$ a compact Lie algebra. Under these conditions we showed, at the beginning of this Section, that there exists $Y_0 \in \mathcal{Z}(\mathfrak{g})$ satisfying that $\phi_0(Y_0) = 1$. Then, using Theorems 6.2 and 6.3, we deduce the following result.

THEOREM 6.4: *Let $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, X_0))$ be a generalized Lie bialgebra, with $\phi_0 \neq 0$ and $\mathfrak{g}$ a compact Lie algebra. If $X_0 = 0$ (respectively, $X_0 \neq 0$) then it is of the first kind (respectively, the second or third kind).*

*(b) The case $\phi_0 = 0$*

We will describe the structure of a generalized Lie bialgebra $((\mathfrak{g}, 0), (\mathfrak{g}^*, X_0))$, $\mathfrak{g}$ being a compact Lie algebra and $X_0 \neq 0$. First, we will examine a suitable example.

Let $(\mathfrak{h}, \mathfrak{h}^*)$ be a Lie bialgebra and $\Psi$ be an endomorphism of $\mathfrak{h}$, $\Psi\colon \mathfrak{h} \to \mathfrak{h}$. Assume that $\Psi$ is a 1-cocycle of $\mathfrak{h}$ with respect to the adjoint representation

$ad^{\mathfrak{h}}: \mathfrak{h} \times \mathfrak{h} \to \mathfrak{h}$ and that $\Psi^* - Id$ is a 1-cocycle of $\mathfrak{h}^*$ with respect to the adjoint representation $ad^{\mathfrak{h}^*}: \mathfrak{h}^* \times \mathfrak{h}^* \to \mathfrak{h}^*$. Here, $\Psi^*: \mathfrak{h}^* \to \mathfrak{h}^*$ is the adjoint linear map of $\Psi: \mathfrak{h} \to \mathfrak{h}$. Denote by $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}$ the direct product of the Lie algebras $\mathfrak{h}$ and $\mathbb{R}$ and consider on $\mathfrak{g}^* \cong \mathfrak{h}^* \oplus \mathbb{R}$ the Lie bracket $[,]^{\mathfrak{g}^*}$ defined by

$$[(\alpha,\lambda),(\beta,\mu)]^{\mathfrak{g}^*} = ([\alpha,\beta]^{\mathfrak{h}^*} - \lambda(\Psi^* - Id)(\beta) + \mu(\Psi^* - Id)(\alpha), 0), \tag{6.14}$$

for $(\alpha,\lambda),(\beta,\mu) \in \mathfrak{h}^* \oplus \mathbb{R} \cong \mathfrak{g}^*$. Using (6.14), that $(\mathfrak{h},\mathfrak{h}^*)$ is a Lie bialgebra and the fact that $\Psi$ is a 1-cocycle, we deduce that (3.1) holds. Thus, the pair $((\mathfrak{g},0),(\mathfrak{g}^*,(0,1)))$ is a generalized Lie bialgebra. Moreover, it is clear that if $\mathfrak{h}$ is a compact Lie algebra then $\mathfrak{g}$ is also compact.

Next, suppose that $\mathfrak{h}$ is compact and semisimple and denote by $d_{\mathfrak{h}^*}$ the Chevalley–Eilenberg differential of $\mathfrak{h}^*$. Then, from Lemma 3.4, it follows that there exist $r \in \wedge^2\mathfrak{h}$ and $Z \in \mathfrak{h}$ such that

$$d_{\mathfrak{h}^*}X = -[X,r]^{\mathfrak{h}}, \quad \Psi(X) = [X,Z]^{\mathfrak{h}}, \quad \Psi^*(\alpha) = coad^{\mathfrak{h}}_Z\alpha = \mathcal{L}_Z\alpha, \tag{6.15}$$

for $X \in \mathfrak{h}$ and $\alpha \in \mathfrak{h}^*$, where $coad^{\mathfrak{h}}: \mathfrak{h} \times \mathfrak{h}^* \to \mathfrak{h}^*$ is the coadjoint representation. Using (6.15) and the fact that $\Psi^* - Id$ is an adjoint 1-cocycle of $\mathfrak{h}^*$, we deduce that

$$([[X,Z]^{\mathfrak{h}},r]^{\mathfrak{h}} + [Z,[X,r]^{\mathfrak{h}}]^{\mathfrak{h}})(\alpha,\beta) = (d_{\mathfrak{h}^*}X)(\alpha,\beta) = -[X,r]^{\mathfrak{h}}(\alpha,\beta),$$

for $\alpha,\beta \in \mathfrak{h}^*$. Thus, the equality $[X,[Z,r]^{\mathfrak{h}}]^{\mathfrak{h}} = [[X,Z]^{\mathfrak{h}},r]^{\mathfrak{h}} + [Z,[X,r]^{\mathfrak{h}}]^{\mathfrak{h}}$ implies that

$$[X,[Z,r]^{\mathfrak{h}}]^{\mathfrak{h}} = -[X,r]^{\mathfrak{h}}, \quad \text{for all } X \in \mathfrak{h}. \tag{6.16}$$

The compact character of $\mathfrak{h}$ allows us to choose an $ad^{\mathfrak{h}}$-invariant scalar product $<,>$ on $\mathfrak{h}$. We will also denote by $<,>$ the natural extension of $<,>$ to $\wedge^2\mathfrak{h}$. This extension is a scalar product on $\wedge^2\mathfrak{h}$ and, in addition, it is easy to prove that $<[X,s]^{\mathfrak{h}},t> = -<s,[X,t]^{\mathfrak{h}}>$, for $X \in \mathfrak{h}$ and $s,t \in \wedge^2\mathfrak{h}$. Thus (see (6.16)),

$$<[Z,r]^{\mathfrak{h}},[Z,r]^{\mathfrak{h}}> = -<r,[Z,[Z,r]^{\mathfrak{h}}]^{\mathfrak{h}}> = <r,[Z,r]^{\mathfrak{h}}> = 0,$$

i.e.,

$$[Z,r]^{\mathfrak{h}} = 0. \tag{6.17}$$

Then, from (6.15), (6.16) and (6.17), we conclude that the Lie bracket $[,]^{\mathfrak{h}^*}$ is trivial.

*Remark 6.5:* If $\mathfrak{h}$ is not semisimple then the Lie bracket $[,]^{\mathfrak{h}^*}$ is not, in general, trivial. In fact, suppose that $\mathcal{Z}(\mathfrak{h}) \neq \{0\}$. We know that $\mathfrak{h}$ is isomorphic, as a Lie algebra, to the direct product $\mathfrak{h}' \oplus \mathcal{Z}(\mathfrak{h})$, where $\mathfrak{h}'$ is a compact semisimple Lie subalgebra of $\mathfrak{h}$. Therefore, if $\Psi\colon \mathfrak{h} \cong \mathfrak{h}' \oplus \mathcal{Z}(\mathfrak{h}) \to \mathfrak{h} \cong \mathfrak{h}' \oplus \mathcal{Z}(\mathfrak{h})$ is the projection on the subspace $\mathcal{Z}(\mathfrak{h})$, it follows that $\Psi$ is an adjoint 1-cocycle of $\mathfrak{h}$. Furthermore, if on $(\mathfrak{h}')^*$ we consider the trivial Lie bracket and on $\mathcal{Z}(\mathfrak{h})^*$ an arbitrary (nontrivial) Lie bracket, then the direct product $(\mathfrak{h}')^* \oplus \mathcal{Z}(\mathfrak{h})^* \cong \mathfrak{h}^*$ is a Lie algebra, the pair $(\mathfrak{h}, \mathfrak{h}^*)$ is a Lie bialgebra and the endomorphism $\Psi^* - Id$ is an adjoint 1-cocycle of $\mathfrak{h}^*$.

Now, we prove

THEOREM 6.6: *Let $((\mathfrak{g}, 0), (\mathfrak{g}^*, X_0))$ be a generalized Lie bialgebra with $X_0 \neq 0$ and $\mathfrak{g}$ a compact Lie algebra. Then:*

(i) *There exists a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $\mathfrak{g}$ is isomorphic, as a Lie algebra, to the direct product $\mathfrak{h} \oplus \mathbb{R}$. Moreover, under the above isomorphism, $\mathfrak{h}^*$ is a Lie subalgebra of $\mathfrak{g}^*$, the pair $(\mathfrak{h}, \mathfrak{h}^*)$ is a Lie bialgebra, $X_0 = (0, 1) \in \mathfrak{h} \oplus \mathbb{R} \cong \mathfrak{g}$ and the Lie bracket $[,]^{\mathfrak{g}^*}$ on $\mathfrak{g}^*$ is given by*

$$[(\alpha, \lambda), (\beta, \mu)]^{\mathfrak{g}^*} = ([\alpha, \beta]^{\mathfrak{h}^*} - \lambda(\Psi^* - Id)(\beta) + \mu(\Psi^* - Id)(\alpha), 0),$$

*where $\Psi \in End(\mathfrak{h})$ is an adjoint 1-cocycle of $\mathfrak{h}$ and $\Psi^* - Id$ is an adjoint 1-cocycle of $\mathfrak{h}^*$.*

(ii) *If $\dim \mathcal{Z}(\mathfrak{g}) = 1$, then the Lie bracket $[,]^{\mathfrak{h}^*}$ is trivial and there exists $Z \in \mathfrak{h}$ such that $\Psi(X) = [X, Z]^{\mathfrak{h}}$, for all $X \in \mathfrak{h}$.*

*Proof:* (i) From (3.3) it follows that $X_0 \in \mathcal{Z}(\mathfrak{g})$. We consider an $ad^{\mathfrak{g}}$-invariant scalar product $<,>$ on $\mathfrak{g}$ and the 1-form $\theta_0 \in \mathfrak{g}^*$ defined by $\theta_0(X) = < X, X_0 >$, for all $X \in \mathfrak{g}$. We have that $\theta_0$ is a 1-cocycle of $\mathfrak{g}$ and we can assume, without the loss of generality, that $\theta_0(X_0) = 1$. Then, using (3.1) and the fact that $X_0$ is a 1-cocycle of $\mathfrak{g}^*$, we deduce that the Lie subalgebra $\mathfrak{h}$ is the annihilator of the subspace generated by $\theta_0$ and that the endomorphism $\Psi\colon \mathfrak{h} \to \mathfrak{h}$ is given by $\Psi(X) = X - i(\theta_0)(d_* X)$, where $d_*$ is the Chevalley–Eilenberg differential of $\mathfrak{g}^*$.

(ii) If $dim \mathcal{Z}(\mathfrak{g}) = 1$ then $\mathfrak{h}$ is compact and semisimple and the result follows. ∎

## Appendix A. Algebraic Jacobi structures

In this Appendix, we will deal with an algebraic version of the concept of Jacobi structure.

*Definition A.1:* Let $(\mathfrak{g}, [\,,]^{\mathfrak{g}})$ be a real Lie algebra of finite dimension. An algebraic Jacobi structure on $\mathfrak{g}$ is a pair $(r, X_0)$, with $r \in \wedge^2\mathfrak{g}$ and $X_0 \in \mathfrak{g}$ satisfying

$$[r,r]^{\mathfrak{g}} = 2X_0 \wedge r, \quad [X_0, r]^{\mathfrak{g}} = 0,$$

where $[\,,]^{\mathfrak{g}}$ is the algebraic Schouten bracket.

Note that the algebraic Poisson structures on $\mathfrak{g}$ or, in other words, the solutions of the classical Yang-Baxter equation on $\mathfrak{g}$ are just the algebraic Jacobi structures $(r, X_0)$ such that $X_0$ is zero.

Let $G$ be a connected Lie group with Lie algebra $\mathfrak{g}$. Since $[\bar{s}, \bar{t}] = \overline{[s,t]^{\mathfrak{g}}}$, for $s, t \in \wedge^*\mathfrak{g}$, the pair $(r, X_0)$ is an algebraic Jacobi structure on $\mathfrak{g}$ if and only if $(\bar{r}, \bar{X}_0)$ is a left invariant Jacobi structure on $G$.

*Examples A.2:* (i) Let $(\mathfrak{g}, [\,,]^{\mathfrak{g}})$ be a real Lie algebra of odd dimension $2k+1$. We say that $\eta \in \mathfrak{g}^*$ is an **algebraic contact 1-form** on $\mathfrak{g}$ if

$$\eta \wedge (d\eta)^k = \eta \wedge d\eta \wedge \overset{(k}{\ldots} \wedge d\eta \neq 0,$$

where $d$ is the Chevalley–Eilenberg differential of $\mathfrak{g}$ (see [7]). In such a case, $(\mathfrak{g}, \eta)$ is termed a **contact Lie algebra**. If $(\mathfrak{g}, \eta)$ is a contact Lie algebra, we define $r \in \wedge^2\mathfrak{g}$ and $X_0 \in \mathfrak{g}$ as follows,

$$r(\alpha, \beta) = d\eta(\flat_\eta^{-1}(\alpha), \flat_\eta^{-1}(\beta)), \quad X_0 = \flat_\eta^{-1}(\eta), \tag{A.1}$$

for $\alpha, \beta \in \mathfrak{g}^*$, where $\flat_\eta \colon \mathfrak{g} \to \mathfrak{g}^*$ is the isomorphism of vector spaces given by

$$\flat_\eta(X) = i(X)(d\eta) + \eta(X)\eta, \tag{A.2}$$

for $X \in \mathfrak{g}$. The vector $X_0$ is the **Reeb vector** of $\mathfrak{g}$ and it is characterized by the relations

$$i(X_0)(d\eta) = 0, \quad \eta(X_0) = 1. \tag{A.3}$$

If $G$ is a connected Lie group with Lie algebra $\mathfrak{g}$ then it is clear that the left invariant 1-form $\bar{\eta}$ on $G$ satisfying $\bar{\eta}(\mathfrak{e}) = \eta$ is a contact 1-form. Moreover, the pair $(\bar{r}, \bar{X}_0)$ is just the Jacobi structure on $G$ associated with $\bar{\eta}$ (see, for instance, [6, 12, 25]). Therefore, we deduce that $(r, X_0)$ is an algebraic Jacobi structure on $\mathfrak{g}$.

Using (A.1), (A.2) and (A.3), we find that $\#_r(\alpha) = -\flat_\eta^{-1}(\alpha) + \alpha(X_0)X_0$, for $\alpha \in \mathfrak{g}^*$.

(ii) Let $(\mathfrak{g}, [,]^{\mathfrak{g}})$ be a real Lie algebra of even dimension $2k$. An **algebraic locally conformal symplectic (l.c.s.) structure** on $\mathfrak{g}$ is a pair $(\Omega, \omega)$, where $\Omega \in \wedge^2\mathfrak{g}^*$, $\omega \in \mathfrak{g}^*$ and

$$\Omega^k = \Omega\wedge \overset{(k}{\ldots} \wedge\Omega \neq 0, \quad d\Omega = \omega \wedge \Omega, \quad d\omega = 0. \tag{A.4}$$

The 1-form $\omega$ is the **Lee 1-form** of the l.c.s. structure.

If $(\Omega, \omega)$ is an algebraic l.c.s. structure on $\mathfrak{g}$, one can define $r \in \wedge^2\mathfrak{g}$ and $X_0 \in \mathfrak{g}$ by

$$r(\alpha, \beta) = \Omega(\flat_\Omega^{-1}(\alpha), \flat_\Omega^{-1}(\beta)), \qquad X_0 = \flat_\Omega^{-1}(\omega), \tag{A.5}$$

for $\alpha, \beta \in \mathfrak{g}^*$, $\flat_\Omega\colon \mathfrak{g} \to \mathfrak{g}^*$ being the isomorphism of vector spaces given by

$$\flat_\Omega(X) = i(X)\Omega, \tag{A.6}$$

for $X \in \mathfrak{g}$. If $G$ is a connected Lie group with Lie algebra $\mathfrak{g}$ then it is clear that the left invariant 2-form $\bar{\Omega}$ defines a locally conformal symplectic structure on $G$. Furthermore, the pair $(\bar{r}, \bar{X}_0)$ is just the Jacobi structure on $G$ associated with $\bar{\Omega}$ (see, for instance, [12, 17]). Consequently, we obtain that $(r, X_0)$ is an algebraic Jacobi structure on $\mathfrak{g}$.

In this case, using (A.5) and (A.6), it follows that $\#_r(\alpha) = -\flat_\Omega^{-1}(\alpha)$, for $\alpha \in \mathfrak{g}^*$. In particular, $\#_r\colon \mathfrak{g}^* \to \mathfrak{g}$ is a linear isomorphism.

It is clear that a real Lie algebra $\mathfrak{g}$ is symplectic in the sense of [26] if and only if $\mathfrak{g}$ is l.c.s. and the Lee 1-form is zero. Moreover, if $\mathfrak{g}$ is a symplectic Lie algebra then the 2-vector $r \in \wedge^2\mathfrak{g}$ given by (A.5) is a solution of the classical Yang–Baxter equation on $\mathfrak{g}$.

Now, we introduce the following definition.

*Definition A.3:* Let $(\mathfrak{g}, [,]^{\mathfrak{g}})$ be a real Lie algebra of dimension $n$ and $(r, X_0)$ be an algebraic Jacobi structure on $\mathfrak{g}$. The rank of $(r, X_0)$ is the dimension of the subspace $\#_r(\mathfrak{g}^*) + < X_0 > \subseteq \mathfrak{g}$.

Equivalently, the rank of $(r, X_0)$ is $2k \leq n$ (respectively, $2k + 1 \leq n$) if the rank of $r$ is $2k$ and $X_0 \wedge r^k = X_0 \wedge r \wedge \overset{(k}{\ldots} \wedge r = 0$ (respectively, $X_0 \wedge r^k \neq 0$).

If $G$ is a connected Lie group with Lie algebra $\mathfrak{g}$ then it is clear that the rank of an algebraic Jacobi structure $(r, X_0)$ on $\mathfrak{g}$ is just the rank of the Jacobi structure $(\bar{r}, \bar{X}_0)$ on $G$. Thus, the rank of a contact Lie algebra (respectively, l.c.s. Lie algebra) of dimension $2k + 1$ (respectively, $2k$) is $2k + 1$ (respectively, $2k$). Conversely, using some well-known results about transitive Jacobi manifolds (see

[6, 12, 17]), one may prove that if $(r, X_0)$ is an algebraic Jacobi structure of rank $2k+1$ (respectively, of rank $2k$) on a Lie algebra $\mathfrak{g}$ of dimension $2k+1$ (respectively, of dimension $2k$) then the structure $(r, X_0)$ is associated to an algebraic contact structure (respectively, an algebraic l.c.s. structure) on $\mathfrak{g}$. Moreover,

PROPOSITION A.4: *Let $(\mathfrak{g}, [\,,]^{\mathfrak{g}})$ be a real Lie algebra of dimension $n$ and $(r, X_0)$ be an algebraic Jacobi structure on $\mathfrak{g}$ of rank $m \leq n$. Then, there exists an $m$-dimensional Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $r \in \wedge^2\mathfrak{h}$, $X_0 \in \mathfrak{h}$, the pair $(r, X_0)$ defines an algebraic Jacobi structure on $\mathfrak{h}$ and:*

(i) *If $m$ is odd, the structure $(r, X_0)$ is associated to an algebraic contact structure on $\mathfrak{h}$.*
(ii) *If $m$ is even, the structure $(r, X_0)$ is associated to an algebraic l.c.s. structure on $\mathfrak{h}$.*

*Proof:* Let $G$ be a connected Lie group with Lie algebra $\mathfrak{g}$ and $(\bar{r}, \bar{X}_0)$ be the corresponding left invariant Jacobi structure on $G$. Denote by $\mathcal{F}$ the characteristic foliation on $G$ associated with the Jacobi structure $(\bar{r}, \bar{X}_0)$, that is (see [6, 12, 17]), for every $g \in G$, $\mathcal{F}_g$ is the subspace of $T_gG$ defined by

$$\mathcal{F}_g = (\#_{\bar{r}})_g(T_g^*G) + < \bar{X}_0(g) > .$$

It is clear that

$$\bar{r}(g) \in \wedge^2\mathcal{F}_g, \quad \mathcal{F}_g = (L_g)_*(\mathcal{F}_{\mathfrak{e}}), \quad \dim \mathcal{F}_g = \dim \mathcal{F}_{\mathfrak{e}} = m,$$

for all $g \in G$. Thus, $\mathfrak{h} = \mathcal{F}_{\mathfrak{e}}$ is an $m$-dimensional Lie subalgebra of $\mathfrak{g}$ satisfying the conclusions of the proposition. ∎

## Appendix B. Compact contact Lie algebras

In [7], Diatta proved that if $G$ is a Lie group which admits a left invariant contact structure and a bi-invariant semi-Riemannian metric, then $G$ is semisimple and thus, from Theorem 5 in [2], he deduced that G is locally isomorphic to $SL(2, \mathbb{R})$ or to $SU(2)$. Therefore, if $\mathfrak{h}$ is a compact Lie algebra endowed with an algebraic contact structure, then $\mathfrak{h}$ is isomorphic to $\mathfrak{su}(2)$. Here, we will give a direct proof of this last assertion, and we will describe all the algebraic contact structures on $\mathfrak{su}(2)$.

PROPOSITION B.1: *Let $\mathfrak{h}$ be a compact Lie algebra of dimension $2k+1$, with $k \geq 1$. Suppose that $(r, X_0)$ is an algebraic Jacobi structure on $\mathfrak{h}$ which is* *associated to an algebraic contact structure. Then, $k = 1$, $\mathfrak{h}$ is isomorphic to $\mathfrak{su}(2)$ and*

$$r = \lambda^1 e_2 \wedge e_3 + \lambda^2 e_3 \wedge e_1 + \lambda^3 e_1 \wedge e_2, \quad X_0 = -(\lambda^1 e_1 + \lambda^2 e_2 + \lambda^3 e_3),$$

*where $(\lambda^1, \lambda^2, \lambda^3) \in \mathbb{R}^3 - \{(0,0,0)\}$ and $\{e_1, e_2, e_3\}$ is a basis of $\mathfrak{h}$ such that*

$$[e_1, e_2]^{\mathfrak{h}} = e_3, \quad [e_3, e_1]^{\mathfrak{h}} = e_2, \quad [e_2, e_3]^{\mathfrak{h}} = e_1.$$

*Proof:* Let $\eta$ be the algebraic contact 1-form on $\mathfrak{h}$ associated with the algebraic Jacobi structure $(r, X_0)$ (see Appendix A). We can consider an *ad*-invariant scalar product $<,> : \mathfrak{h} \times \mathfrak{h} \to \mathbb{R}$ on $\mathfrak{h}$ and the vector $X_\eta \in \mathfrak{h}$ characterized by the relation

$$\text{(B.1)} \qquad \eta(X) = < X, X_\eta >, \quad \text{for } X \in \mathfrak{h}.$$

If $d$ is the Chevalley–Eilenberg differential on $\mathfrak{h}$ then, using (B.1) and the fact that $<,>$ is an *ad*-invariant scalar product, we see that $i(X_\eta)(d\eta) = 0$. This implies that

$$\text{(B.2)} \qquad \mathrm{Ker}(d\eta) = < X_0 > = < X_\eta > .$$

Next, we will prove that the rank of $\mathfrak{h}$ is 1. Assume that there exists $Y \in \mathfrak{h}$ such that $[X_\eta, Y]^{\mathfrak{h}} = 0$. From (B.1), we obtain that

$$(i(Y)d\eta)(X) = - < X_\eta, [Y, X]^{\mathfrak{h}} > = 0, \quad \text{for all } X \in \mathfrak{g}.$$

Thus, using (B.2), we deduce that $X_\eta$ and $Y$ are linearly dependent.

Therefore, $< X_\eta >$ is a maximal abelian subspace of $\mathfrak{h}$. This implies that the rank of $\mathfrak{h}$ is 1 and $\mathfrak{h}$ is isomorphic to $\mathfrak{su}(2)$.

Let $\eta$ be an arbitrary 1-form on $\mathfrak{h}$, $\eta \neq 0$, then $\eta$ is an algebraic contact 1-form. If $\eta = \mu_1 e^1 + \mu_2 e^2 + \mu_3 e^3$, where $\{e^1, e^2, e^3\}$ denotes the dual basis of $\{e_1, e_2, e_3\}$, the algebraic Jacobi structure $(r, X_0)$ associated with $\eta$ is given by (see Appendix A)

$$r = \lambda^1 e_2 \wedge e_3 + \lambda^2 e_3 \wedge e_1 + \lambda^3 e_1 \wedge e_2, \quad X_0 = -(\lambda^1 e_1 + \lambda^2 e_2 + \lambda^3 e_3)$$

with $\lambda^i = -\mu_i/(\mu_1^2 + \mu_2^2 + \mu_3^2)$, for $i \in \{1, 2, 3\}$. ∎

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